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A comparison of two portfolio selection models*

1 INTRODUCTION

This paper presents an empirical examination of two of the more popular mathematical portfolio selection models. The first is the original model proposed by Harry Markowitz[5] and the empirical tests consider the effect on this model of variations in the upper bound (i.e. the maximum proportion of funds to be invested in any one security). The second model considered is Sharpe's[6] Diagonal (or Index) Model which is an extension of the original Markowitz Model but which is computationally far simpler. The results of these two models are compared in an attempt to ascertain differences in their behaviour.

Before presenting the empirical results obtained, a brief résumé of the theory underlying the two portfolio selection techniques is given.

2 THE MODELS AND THE DATA

Both the original Markowitz Model and Sharpe's Diagonal Model assume that there are basically two factors to be considered in choosing a portfolio. These are:

(i) the expected return on the portfolio; and

(ii) the risk associated with this return (i.e. the standard deviation of the return).

As a consequence, Markowitz[5] derived the following definition:

Definition: Efficient portfolio

A portfolio is said to be 'efficient' if it is impossible to obtain a greater expected return without incurring greater risk and it is impossible to obtain smaller risk without decreasing expected return.

Therefore, the problem is to derive the set of all efficient portfolios since, from this set, the investor can choose the single portfolio best suited to his return/risk requirements.

The set of efficient portfolios is obtained as follows:

Minimize

$-\lambda E_p + \sigma_p^2$ for all $\lambda \geq 0$

Subject to

$\sum_{i=1}^{N} X_i = 1$

$X_i \geq 0;\ i = 1, 2, \ldots, N$

plus any other linear equality constraints imposed by the individual investor,

plus $L_i \leq X_i \leq U_i$ for all $i = 1, 2, \ldots, N$;

where

$E_p = \sum_{i=1}^{N} X_i E_i$, and

$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}$, where

N = the number of securities considered

X_i = the proportion of funds invested in the i^{th} security ($i = 1, 2, \ldots, N$)

E_i = the expected return on the i^{th} security ($i = 1, 2, \ldots, N$)

σ_{ij} = covariance between the i^{th} and j^{th} securities ($i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, N$)

E_p = the expected return on the portfolio

σ_p = the standard deviation of the portfolio

U_i = the upper bound on the proportion of funds to be invested in security i; and

L_i = the lower bound.

Since σ_p^2 contains terms of the form X_i^2 and X_iX_j, the above problem is a quadratic programming problem. Algorithms have been proposed for the solution of such problems, the most notable being those of Markowitz,[5] Wolfe,[8] and Sharpe.[7] The solution of these algorithms yield a series of "corner" portfolios which generate the efficient border.

Unfortunately, use of the above model is, in general limited by the large number of estimates required (estimates of all $N(N-1)/2$) distinct covariances are required). To overcome this difficulty, Markowitz[5] suggested that the returns of various securities are related only through their common relationship with some basic underlying factor. Formally, then, the model assumes that the return on security j (R_j) is linearly related to some index I as follows:

$R_j = \alpha_j + \alpha\beta_j I + u_j$

where α_j and β_j are parameters (which must be estimated), and

u_j is a stochastic term with zero mean and variance $\sigma_{u_j}^2$.

If the model further assumes that

$\text{Cov}(u_j, I) = 0$ for all $j = 1, 2, \ldots, N$

and $\text{Cov}(u_i, u_j) = 0$ for all $i \neq j$

(i.e. the model assumes that any two securities are related only through their mutual relationship to the model), then it can be shown that

$E_j = \alpha_j + \beta_j E_I$

$\sigma_j^2 = \beta_j^2 \sigma_I^2 + \sigma_{u_j}^2$

$\sigma_{ij} = \beta_i \beta_j \sigma_I^2$

where E_I = the expected level of the index, and

σ_I^2 = the variance of the index.

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The computations were carried out using the Univac 1106 at the Computer Centre of the University of Cape Town.

Hence, the analyst need only estimate
(i) the parameters α_j, β_j and $\sigma^2_{u_j}$, for each security, and
(ii) E_I and σ^2_I.

Thus, the model requires only 3N + 2 estimates which is considerably less than N(N − 1)/2 for large N.

In addition, Sharpe[6] noticed that if an index is used for this purpose, it becomes unnecessary to multiply out all the entries of the covariance matrix. He found that by setting

$$b_p \sum_{j=1}^{N} X_j\beta_j$$

the model could be written as:

$$\text{Min } -\lambda \left(\sum_{j=1}^{N} X_i\alpha_i + b_pE_I \right) + \left(b_p\sigma^2_I + \sum_{i=1}^{N} X_i\sigma^2_{u_i} \right)$$

for all $\lambda \geq 0$

subject to

$$\sum_{i=1}^{N} X_i\beta_i = b_p$$

$$\sum X_i = 1$$

$$X_i \geq 0$$

plus any other linear constraints or bounds.

Since the only quadratic terms which appear in the above formulation are the squared ones, the covariance matrix has been reduced to a diagonal form and this makes the solution of the problem far simpler. This theory can easily be extended to allow for more than one index.

Before concluding this section, a brief description of the data used in the empirical tests, is given.

A problem which immediately presented itself was that of estimating. There are vast input requirements demanded by the models and these are best obtained from a security analyst. However, one cannot approach such an analyst in 1974 and ask him to estimate the return on share X, say, in 1969. His estimates will obviously be strongly biased towards what has actually happened. So, in order to test these models empirically, one has to resort to estimates based on past prices alone. This method of estimating is generally thought to be rather unsatisfactory. But, in this case there is no alternative and so historical estimates had to be used. Nevertheless, since the results are mainly of a relative nature, they will almost certainly hold if a different form of estimating is used.

Hence, yearly data for the period 1962-1973 were used to provide input for the portfolio selection models. Without delving into the intracies of takeovers, etc., it was found that 175 shares quoted on the J.S.E. in January 1962 were still quoted in December 1973, and therefore the required price and dividend histories for the entire period were available. These 175 shares were taken as the universe of all possible shares.

The yearly return on each security was computed for the period 1962-1973 using the formula

$$R_i(t) = \frac{P_i(t) + D_i(t) - P_i(t-l)}{P_i(t-l)}$$

where
$R_i(t)$ is the return on the i^{th} security in the t^{th} period
$P_i(t)$ is the price of the i^{th} security at the end of the t^{th} period; and
$D_i(t)$ is the total of all dividends paid in the t^{th} period.

3 THE MARKOWITZ[5] MODEL

In this section the effect of different upper bounds (on the amount to be invested in any one security) on the original Markowitz model is examined from an empirical point of view. Since the option not to invest in a given security should always be allowed for, there is no need to vary the lower bound — the obvious lower bound of zero will be applicable in almost every case.

Varying the upper bound will logically have a far greater effect and a low upper bound has been favoured in the literature for two main reasons.

(i) Decreasing the upper bound means that, of necessity, more and more securities must be included in each portfolio. Thus, the inclusion of an upper bound which is not too high can be used to enforce diversification.

(ii) If the total amount to be invested is very large, then the imposition of an upper bound will increase the feasibility of the model since it will ensure that the proportion of funds to be invested in any one security will be a realistic amount and not too large for practical application.

In using Markowitz's model, as was mentioned in the previous section, a large number of estimates are required. If all 175 shares were considered this would be an enormous problem, even for a high speed computer. Therefore, a randum sample of 50 shares was chosen from the universe of 175 shares and the model was empirically tested using only these 50 shares. A list of these randomly chosen shares may be found in Table A.1 of the Appendix. Thus, the expected return, the variance of this return and the covariance of each pair of securities, was estimated for these 50 shares using the data of the initial period (1962-1970).

To test the effect of the variation of the upper bound on the efficient set, the problem was solved using five different sets of bounds:

(i) $0{,}0 \leq X_i \leq 1{,}00$
(ii) $0{,}0 \leq X_i \leq 0{,}50$
(iii) $0{,}0 \leq X_i \leq 0{,}25$
(iv) $0{,}0 \leq X_i \leq 0{,}10$
(v) $0{,}0 \leq X_i \leq 0{,}05$

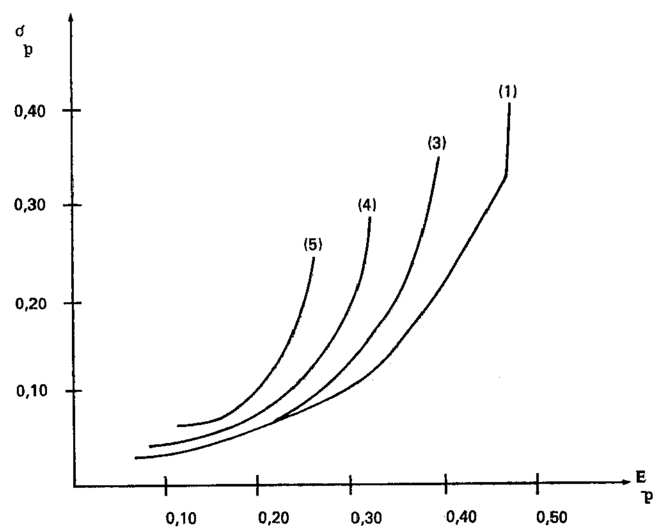

Figure 1

The results of these tests are best shown graphically as in Figure 1. In this figure, the actual efficient set for each case except (ii) is sketched. Thus, the graph marked (1) indicates the efficient set for an upper bound of 1,0; (3) indicates that for an upper bound of 0,25; (4) that of 0,10; and (5) that of 0,05. The reason for not including (ii) was that except for the first three (out of twenty six) corner portfolios, this case produced the identical efficient set as (1).

In order to see the differences in these four cases more clearly, the following table (Table 1) was constructed:

Table 1

Graph	Upper bound	Minimum number of shares held	Number of corner portfolios	E_p	Number of shares held	Time
(1)	1,00	1	26	0,29	6	1:55
(3)	0,25	4	52	0,27	9	3:47
(4)	0,10	10	86	0,23	14	5:46
(5)	0,50	20	90	0,20	22	6:12

As can be seen from Figure 1, the higher upper bound efficient sets dominate the lower upper bound sets; that is (1) dominates (3) which dominates (4) which dominates (5). Most simply, this means that for the same level of risk, the higher the upper bound, the greater the expected return on the portfolio will be. This is illustrated by columns five and six of Table 1, which present results showing the expected return and the number of shares held in the portfolio which would be chosen by an efficient investor willing to accept a level of risk (that is, standard deviation) of 0,1 (i.e. 10%). Clearly, as the upper bound decreases so does the expected return (column 5) on the efficient portfolio for that level of risk.

What is more, the introduction of a lower upper bound causes far more corner portfolios to be generated (column 4 of Table 1) and this results in far more computer time being required.

Thus, the empirical tests presented above suggest that the proposed advantages of a low upper bound ((i) diversification and (ii) feasibility) be reconsidered. Now, except for the first six corner portfolios, all of which are very high risk portfolios, no individual security ever attracted more than forty percent of the total funds even when the upper bound was its maximum, 1,0. In fact, the proportions were usually below thirty percent. In addition, except for these six initial corner portfolios, every portfolio contained at least six different securities which indicates that the model tends to produce diversification anyway. Moreover, it has been shown from studies on The New York Stock Exchange (e.g. Sharpe[7], Fisher and Lorie[4]), that the effect of diversification is minimal, once more than a certain number of securities (usually about ten) have been included in the portfolio. In fact, as Fisher and Lorie[4] show:

"Portfolios containing eight stocks have frequency distributions strikingly similar to those of portfolios containing larger numbers of stocks, except for tails beyond the fifth and ninetieth centiles."

Clearly then, the above arguments rather negate the diversification argument in favour of an upper bound other than 1,0.

Unfortunately, the second point, concerning the feasibility of proportions, cannot be as easily dismissed. But, including a low upper bound does have the effect of increasing the computer costs involved and any practical investor using this approach should be aware of the possible dangers of this. Nevertheless, if the amount to be invested is very large, an upper bound of 1,0 may provide an unpractical solution. Thus, each problem will have to be solved individually, depending on its particular characteristics.

The overall conclusion which may be made from this aspect of the study is that a low upper bound should be applied if and only if the total amount to be invested is very large. For many practical cases, the maximum allowable upper bound of 1,0 will produce the best results, with the model itself ensuring diversification.

4 THE INDEX MODELS

In this section, the index models proposed by Sharpe[6-7] are empirically examined. Specifically, three types of models (one-index, two-index and five-index) are considered, and the results are compared and contrasted with those of the previous section. But before presenting the results, a brief description of the data used is given.

Once again (as in the previous section), the portfolio is chosen from only a random sample of 50 shares and not the entire 175 share universe. Thus, for the purpose of comparison, the same random sample as before (cf. Table A.1 of the Appendix) is used. However, all 175 securities are used in the construction of the various indices as is discussed below.

As suggested by Cohen and Fitch[1], an aggregate performance index, which is more relevant to the particular universe of 175 shares used, was constructed rather than using any of the standard published indicies. This index is the unweighted average of the return on all securities in the universe and was constructed for each of the years 1962 to 1970. The arithmetic average of the actual levels of this index for the nine years 1962-1970 was then assumed to be an estimate of the expected level of the index for the period 1971-1973. Similarly, an estimate of the standard deviation of the index for the latter period, was calculated using the level of the index in the nine previous years. Also, since the return on each security as well as the level of the index are known for the nine year period 1962-1970, estimates of β_i and $\sigma^2_{u_i}$

$$R_i = \alpha_i + \beta_i I + u_i$$

can be obtained by regressing R_i on I, the level of the index. These estimates were then used as input for the model.

For the two-index case, the universe of 175 securities was broken into what was felt to be two distinct classes, one containing all the mining shares (65) and the other all industrial shares (110). Indices of these two sections were then constructed in exactly the same manner as the aggregate index above. In addition, the covariance between the two indices was estimated. Once again, estimates of the β_{i1}, β_{i2}, and $\sigma^2_{u_1}$ in the model

$$R_i = \alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + u_i$$

can be computed using regression techniques.

Finally, for the five-index problem, the universe of 175 securities was divided into five distinct groups as follows:

I_1 = Coal index (16 shares)
I_2 = Gold index (35 shares)
I_3 = Other minerals index (14 shares)
I_4 = Financial mining and industrial index (28 shares)
I_5 = Miscellaneous index (the remaining 82 shares)

Admittedly, this might not be an ideal subdivision but with only nine years of data available, it was felt that this was the maximum subdivision allowable. As before, indices of these five sections were constructed by computing the arithmetic average of the returns on all securities included in the respective subdivisions. In addition, the covariance between each pair of indices was estimated and estimates of the parameters β_{i1}, β_{i2}, . . ., β_{i5} and $\sigma^2_{u_1}$ in the model

$$R_i = \alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \ldots + \beta_{i5}I_5 + u_i$$

were again calculated using regression techniques.

The algorithm used to solve these problems was that proposed by Sharpe[7] and it should be noted that an upper bound of 1,0 was assumed to be applicable in all cases.

Using the methods described above, the various input data were collected and the programme was used to construct the efficient set of portfolios for each of the three models. These efficient sets were sketched and are presented in Figure 2. In addition, the Markowitz efficient set, obtained in the previous section, was also sketched to enable a visual comparison. The results appear to be reasonably satisfactory, with the Markowitz approach, since it is the most exact, dominating the others. In addition, the five-index model dominates the two-index model which in turn almost dominates the one-index model. This is also to be expected since the more indices included, the more realistic the model should be. However, there is clearly a difference between the results obtained using the Markowitz approach, and those obtained using the index models. Clearly then, a closer examination is called for.

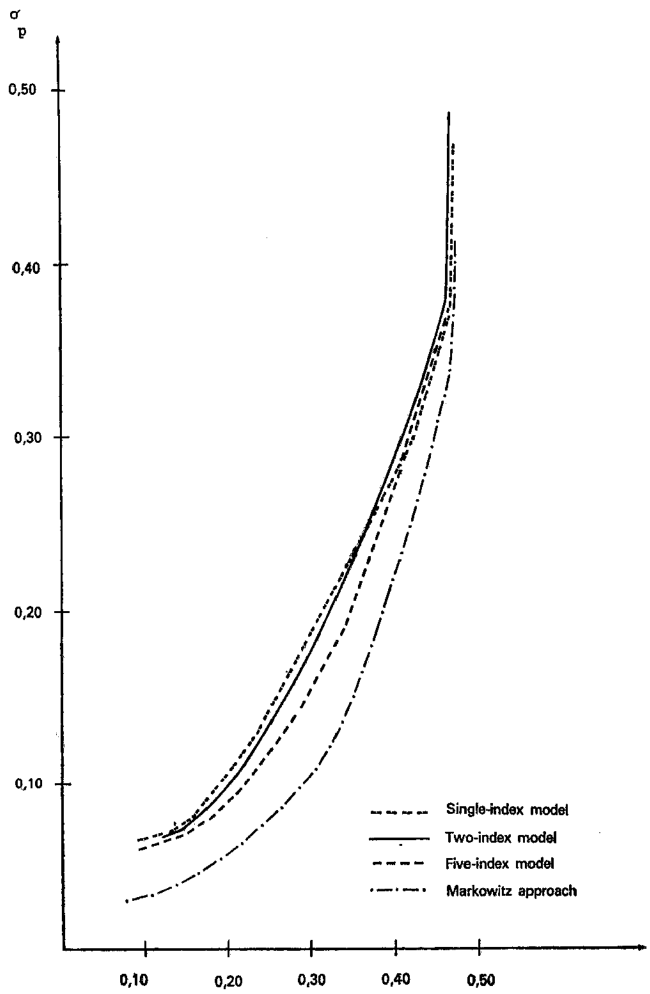

Figure 2

In order to do this, the portfolio a typical investor might select was chosen, using each of the four models. To do this, Farrar's[3] coefficient of risk aversion was used and was assumed to be 0,08969. The actual portfolios chosen are listed in Table 2.

TABLE 2

% held in portfolio

Share	1-Index model	2-Index model	3-Index model	Markowitz model
Apex Mines	4,39	2,41		2,31
Natal Ants	6,12	7,60	6,14	
S.A. Coal	8,37	5,82	6,93	
Witbank	2,37			
De Beers	0,38			14,07
Leslie			5,43	
Winkels	4,95	4,76		29,63
Buffels				9,01
Vaal Reefs	3,95		3,36	
Loraine	1,36		37,48	
St. Helena	8,78	20,64		
Doornfontein	7,09	7,13		
Libanon	7,79	8,07		
Western Deep	9,06	3,27		
P.P. Rust.	2,60	3,11	15,89	
Cons. Murch.	8,01	8,93	9,25	35,70
New Wits.	15,06	9,89		
Trade & Ind.	1,85	1,37	3,47	
Natal Chem.	3,62	2,59	1,52	
Eriksen	0,22	1,03		
C.N.A.	2,00	4,37	4,57	
Truworths				6,59
Gledhow	1,42	7,31		2,68
Reynolds	0,43	1,70	5,96	
E_p (in %)	16,87	17,89	20,18	27,40
σ_p	8,69	8,51	8,89	9,23

As can be seen, there are great differences in the results obtained. This is most easily noticed by considering the number of shares held in each portfolio. The portfolio for the 1-index model contains 21 different shares, the 2-index model 17, the 5-index model 11 and the Markowitz approach only 7. Similarly, the expected return improves as one moves from left to right across the table, while the standard deviation remains almost constant. It is almost impossible to decide exactly what causes this difference but it is probably the estimates of the individual standard deviations. The index models appear consistently to overestimate the standard deviation, and this results in the selection of a portfolio which is far too diversified and is thus not truly representative of the investor's needs.

Since the effect of a low upper bound also has the effect of forcing wide diversification, it should be noted that the imposition of such a bound will cause the index models to be far closer to the Markowitz approach than in the unbounded case considered above. This is further borne out by the fact that in the portfolios chosen from the one and two index models, the maximum amount invested in any one security was just over 20%.

5 CONCLUSIONS

As was mentioned in the previous section, these index models assume that securities are related only through their common indices. If enough indices are included, then clearly the model will produce an almost perfect representation. In fact, taken to the extreme, each index can be made to comprise exactly one security. Then, in the problem discussed above, there will be 175 indices and the covariance between each index will merely be the covariance between the individual securities. Thus, the problem would reduce to the basic Markowitz approach. However, as mentioned previously, this approach requires a substantial amount of input and thus might not be feasible for practical applications. Clearly, some trade-off between the number of indices included and the amount of input required is necessary. Unfortunately, no universal rule can be established and each case must be considered on its own merits.

In addition, the choice of the indices used will be crucial and hence this aspect would require a very detailed study. Nevertheless, it is very unlikely that any single index will ever provide satisfactory results unless a very low upper bound is applied. It should be noted that the many studies on The New York Stock Exchange (for example, Sharpe[6] and Cohen and Pogue[2]) indicating that the one index model provides an appropriate approximation to the Markowitz approach, all used an extremely low upper bound — e.g. in the case of Cohen and Pogue[2], an upper bound of 0,05 (i.e. 5%) was used.

From the results presented in this section it may be concluded that the Markowitz approach produces results which are significantly superior to those obtained using an Index model. Thus, in practice, the investor wishing to use a risk-return approach to portfolio selection should strive to apply the basic Markowitz formulation. If this is impossible, an index model may be used, but it is stressed that the results obtained may be overly conservative. However, if the total amount to be invested is very large, thus forcing a low upper bound to be imposed on the amount invested in any one security, then the index models may be used with much more confidence.

APPENDIX TABLE A. 1

Number	Share
1	Apex Mines Ltd.
2	Natal Anthracite Colliery Ltd.
3	South African Coal Estates (Witbank) Ltd.
4	Witbank Colliery Ltd.
5	De Beers Consolidated Mines Ltd.
6	The Grootvlei Proprietary Mines Ltd.
7	West Rand Consolidated Mines Ltd.
8	Leslie Gold Mines Ltd.
9	Winkelhaak Mines Ltd.
10	Buffelsfontein Gold Mining Co. Ltd.
11	Stilfontein Gold Mining Co. Ltd.
12	Vaal Reefs Exploration & Mining Co. Ltd.
13	Loraine Gold Mines Ltd.
14	St. Helena Gold Mines Ltd.
15	Doornfontein Gold Mining Co. Ltd.
16	Libanon Gold Mining Co. Ltd.
17	Western Deep Levels Ltd.
18	Potgietersrust Platinums Ltd.
19	Consolidated Murchison Ltd.
20	Free State Development & Investment Corporation Ltd.
21	General Mining & Finance Corporation Ltd.
22	Johannesburg Consolidated Investment Co.
23	New Witwatersrand Gold Exploration Co Ltd
24	Union Corporation Ltd.
25	Bonuskor Bpk.
26	De Beers Industrial Corporation Ltd.
27	Trade & Industry Acceptance Corporation Ltd
28	The Common Fund Investment Society Ltd.
29	Federale Beleggingskorporasie Bpk.
30	Cape Portland Cement Co. Ltd.
31	The African Clothing Factory (Ensign) Ltd.
32	Weil and Aschheim (Holdings) Ltd.
33	Natal Chemical Syndicate Ltd.
34	Sea Products (S.W.A.) Ltd.
35	The Premier Milling Co. Ltd.
36	Stein Brothers (Holdings) Ltd.
37	International Combustion (Africa) Ltd.
38	Stewarts & Lloyds of South Africa Ltd.
39	The Union Steel Corporation of South Africa Ltd.
40	Eriksen Consolidated Holdings Ltd.
41	Premier Paper Mills Ltd.
42	Argus Printing & Publishing Co. Ltd.
43	C.N.A. Investments Ltd.
44	Stuttaford and Company Ltd.
45	Truworths Ltd.
46	Woolworths Holdings Ltd.
47	Gledhow Sugar Co. Ltd.
48	Reynolds Brothers Ltd.
49	Consolidated Textile Mills Investment Corporation Ltd.
50	Rembrandt Group Ltd.

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