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On calculating average rates of return

Assume you invested R500 for two years. At the end of the first year it had risen to R1 000, representing a rate of return (ROR) for the year of 100%. At the end of the second year the R1 000 had dropped back to R500, representing a ROR for the year of —50%. What was the average ROR?

The answer to this question depends on *which* average ROR is sought. It could be any one of the following:

arithmetic mean return
time-weighted return
rand-weighted return.

In the above example, the three values are 25% p.a., 0% p.a. and 0% p.a., respectively. Clearly, the arithmetic mean of 25% p.a. is misleading since if you invested R500 and ended up with R500 (with no interim withdrawals) your true average ROR was zero.

The purpose of this paper is to examine the above three ROR concepts. In doing so, we assume an investment which yielded the following annual rates of return over a four-year period:

year 1 : 10%
year 2 : 3%
year 3 : 15%
year 4 : 20%

Unless otherwise stated, we assume *annual* compounding/discounting throughout.[1]

ARITHMETIC MEAN ROR

In our four-year example, the arithmetic mean is 12% p.a. but as already indicated this is misleading. However, it is not completely meaningless since it "can be thought of as the mean value of the withdrawals (expressed as a fraction of the initial portfolio value) that can be made at the end of each interval while maintaining the principal intact".[2]

In the example, the investor can withdraw 10% of the initial investment at the end of year 1, 3% at the end of year 2, and so on, without reducing the money value of his initial investment. In the event of the return in any year being negative, he would have to *add* that amount to his initial investment in order to maintain its money value.

Despite this meaning which can be given to the arithmetic mean ROR, the measure is not often used by experts since the other two measures are, generally speaking, superior.

TIME-WEIGHT ROR

This is a geometric mean approach, viz:

$$[(1+r_1)\,(1+r_2)\ldots.(1+r_n)]^{1/n}-1$$

$$=\left[\prod_{t=1}^{n}(1+r_t)\right]^{1/n}-1$$

where r_1 is the ROR in period 1, r_2 the ROR in period 2, etc., expressed in decimals.

In this approach, the geometric mean[3] of the (1+r) terms[4] is calculated and then 1 is subtracted. In our four-year example, the time-weighted ROR is r_g in:

$$\begin{aligned} r_g &= [(1+r_1)\,(1+r_2)\,(1+r_3)\,(1+r_4)]^{\frac{1}{4}}-1\\ &= [(1{,}10)\,(1{,}03)\,(1{,}15)\,(1{,}20)]^{\frac{1}{4}}-1\\ &= 0{,}1182\\ &= 11{,}82\%\text{ p.a.}\end{aligned}$$

In general, the arithmetic mean ROR and the time-weighted ROR do not coincide. This is because the latter assumes that each period's return is reinvested, whereas the former makes no assumptions with regard to the reinvestment or withdrawal of interim returns.[5]

The arithmetic mean ROR always exceeds the time-weighted ROR except in the special case where the returns in each period are the same, in which event the averages are identical.[6]

Logarithmic transformations are often used to expedite computation of the time-weighted ROR.[7] For instance, the annual rates in our example could be converted from their annual compounding basis to a continuous compounding basis, the arithmetic mean calculated and converted back to an annual compounding basis.[8] Thus:

Assumed annual rate	Equivalent continuous rate
10%	9,531%
3%	2,956%
15%	13,976%
20%	18,232%

The arithmetic mean of the equivalent continuous rates is 11,1738% which becomes 11,82% when converted to an annual compounding basis.

This is a logarithmic approach using Naperian or natural logarithms (base e).[9]

RAND-WEIGHTED ROR

Commonly known as the internal rate of return, this measures the rate of growth of all funds invested in the project during the evaluation period, i.e. the rate of growth of the initial investment plus any reinvestments and minus any withdrawals.

The rand-weighted ROR will be the same as the time-weighted ROR where all interim returns are reinvested. Thus, reverting to our four-year example and assuming an initial investment of R100, where each year's return has been reinvested in the same project, the following net cash flow emerges:

	year 0	year 1	year 2	year 3	year 4
1	—100,00	+110,00	+113,30	+130,30	+156,35
2		—110,00	—113,30	—130,30	
3	—100,00	0	0	0	+156,35

Row 1 shows how the value of the investment increases at the respective annual rates of 10%, 3%, 15% and 20%. Row 2 shows the 'reinvestment' of the original investment plus the annual returns to date. Row 3 shows the net cash flow.

The rand-weighted ROR of the project is then r_i in:

$$100{,}00 = \frac{156{,}35}{(1+r_i)^4}$$

The only positive value of r_i which satisfies this is 0,1182, or 11,82% p.a., which is the same as the time-weighted ROR (r_g).

Alternatively, however, if we assume that each year's return had been withdrawn from the investment, then r_i becomes 11,45% p.a. while r_g stays constant at 11,82% p.a.

This 11,45% is the value of r_i which satisfies the following:[10]

$$100 = \frac{10}{1+r_i} + \frac{3}{(1+r_i)^2} + \frac{15}{(1+r_i)^3} + \frac{120}{(1+r_i)^4}$$

TIME-WEIGHTED vs RAND-WEIGHTED ROR

The rand-weighted ROR (r_i) is not useful in comparing the investment performance of managers of mutual funds and similar organisations. This is because r_i is influenced by how much money remains invested in each fund, and this is normally beyond the control of the managers. The time-weighted ROR (r_g) is more appropriate in such cases.

Our example illustrates this. If the two alternatives in the preceding section represent two different mutual funds, each starting with R100, r_i shows the first fund (where annual returns are reinvested in the fund) to be slightly superior to the second (where annual returns are withdrawn from the fund), viz 11,82% p.a. as against 11,45% p.a. Yet the returns earned each year *on the monies still in the fund* are identical (10%, 3%, 15% and 20%).

If the decision to reinvest or to withdraw interim returns is beyond the control of the managers concerned, then clearly r_g is the appropriate measure and, at 11,82% p.a., is the same for both funds.

Nevertheless, in other situations such as screening decisions in capital budgeting, r_i is more appropriate than r_g, although it can be subject to severe limitations where the net cash flow has more than one change of sign.[11]

LIMITATIONS OF THE TIME-WEIGHTED ROR

This measure cannot be calculated under either of the following conditions:

1 If the ROR in any period is exactly —100%, one of the terms to be multiplied (i.e. one of the link relatives) becomes zero, with the result that the geometric mean cannot be calculated.

2 If the ROR in any period is lower than —100%, it will not be possible to calculate the geometric mean if (a) an odd number of terms to be multiplied is negative (e.g. the ROR in 3 years is below —100%) *and* the number of terms to be multiplied is even (e.g. the project spans 6 years). In this event, it will not be possible to extract a root.

However, if the ROR in any year is negative but not —100% or lower, e.g. if it is —20%, no problem arises in calculating the time-weighted ROR.

CONCLUSION

Of the three average ROR concepts discussed above, the arithmetic mean ROR is probably the least useful and most likely to mislead. The other two (the time-weighted and the rand-weighted rates of return) have more practical relevance. Which of these two is the better depends on the context in which it is being used.

Footnotes

1 In other words, R100 invested at 10% for one year will be worth R110,00 at the end of the year. In biannual compounding, the R100 would be worth R100,00 (1,05) = R105,00 at the end of 6 months and R105,00 (1,05) = R110,25 at the end of 12 months. The greater the frequency of compounding during the year, the greater will be the sum at the end of the year. The limit is found in continuous compounding which, at 10% p.a. (nominal) would convert R100 at the beginning to R110,52 at the end of the year.

2 Modigliani, F. and Pogue, G. A. "An Introduction to Risk and Return", *Financial Analysts Journal*, March-April 1974, p. 69.

3 For n terms, the geometric mean is the nth root of the product of the terms.

4 These terms are called 'link relatives' or 'value relatives'.

5 This does not mean that the arithmetic mean ROR cannot be *conceptualised* in terms of withdrawals (see preceding section).

6 Modigliani and Pogue, *op. cit.* p. 69. For a detailed study of the relationship between the time-weighted ROR and the arithmetic mean, see Young, W. E. and Trent, R. H. "Geometric Mean Approximations of Individual Security and Portfolio Performance", *Journal of Financial and Quantitative Analysis*, June 1969.

7 See Francis, J. C. "Investments: Analysis and Management", McGraw-Hill, 2nd edition, 1976, pp. 685-6.

8 Lorie, J. H. and Hamilton, M. T. "The Stock Market: Theories and Evidence", Irwin, 1973, p. 29.

9 Symbolically the approach can be shown as:

$$rg = \left[\text{antilog}\left(\frac{1}{n}\sum^{n}\log(1+r_t)\right)\right] - 1.$$

10 The net cash flow is obtained as follows, where rows (1), (2) and (3) have the same meaning as before:

	year 0	year 1	year 2	year 3	year 4
1	—100	+110	+103	+115	+120
2		—100	—100	—100	
3	—100	+10	+3	+15	+120

11 These limitations – multiple rates of return, impossibility of calculation, etc. – are well known and need not be restated here.