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Valuation of fixed interest securities

"The time has come," the broker said,
"To speak of many things,
Of yields and turns and semi-gilts,
Of municipals and swings.
And why the price has int'rest too,
And whether rates are keen"

– Apologies to Lewis Carrol

1 INTRODUCTION

In South Africa, trading in fixed interest securities takes place on the basis of yields to redemption. The parties agree on the yield rate and compute the price to be paid for the security involved. (In some overseas countries, the price is agreed upon and the parties compute the yield for their own edification.) It is hoped that this paper will provide sufficient background for the reader to compute stock prices and to understand the underlying philosophy.

Fixed interest-bearing securities, such as Escom and RSA stock, can be valued by the use of compound interest theory. By a process of discounting to the date of settlement the annuity constituted by the stream of coupons remaining to be paid, one obtains their present value (PV). Addition of this figure to the present value of the terminal capital repayment (usually R100%), one arrives at the price, or valuation, of the security in question.

In this paper a formula will be derived for use in the valuation of fixed interest securities. To achieve this, the concept of compound interest and its converse, viz (compound) discounting, will be briefly discussed. These ideas will then be applied to the case of a security paying coupons at six-monthly intervals.

Finally, the paper will deal with the practical question of clean prices and accrued interest.

2 THEORY OF COMPOUND INTEREST

For the purpose of this paper, it will be assumed that interest rates remain unchanged over the period being considered. The assumption does not detract from, nor invalidate, the argument which follows; it merely simplifies the mathematics.

Consider the investment of a sum of money which will grow with the passage of time. Denote this sum as being f(T). After an interval of time denoted by h, the sum will have grown to f(T+h). The growth g of the sum, as a fraction of the original investment, would be calculated as

$$g = \frac{f(T+h) - f(T)}{f(T)}.$$

The *rate* of growth R of the sum would be the growth g divided by the time which has elapsed, i.e.

$$R = \frac{g}{h} = \frac{f(T+h) - f(T)}{h.f(T)} \qquad (1)$$

The result of calculating R as per equation (1) would be the average rate of interest for a period of duration h. In order to ascertain the instantaneous rate of interest it is necessary to reduce h progressively until it is almost zero i.e. as close to zero as possible.

In mathematical parlance, one would evaluate the limit, as h tended to zero, of R. Denoting the instantaneous interest rate by δ, one would write

$$\begin{aligned}\delta &= \lim_{h \to 0} R \\ &= \lim_{h \to 0} \frac{f(T+h) - f(T)}{h.f(T)} \\ &= \frac{1}{f(T)} \cdot \lim_{h \to 0} \frac{f(T+h) - f(T)}{h} \qquad (2)\end{aligned}$$

Because

$$\lim_{h \to 0} \frac{f(T+h) - f(T)}{h} = \frac{df(T)}{dt}$$

Equation (2) can be written as

$$\begin{aligned}&= \frac{1}{f(T)} \cdot \frac{d}{dT} f(T) \\ &= \frac{d}{dT} \ln f(T) \qquad (3)\end{aligned}$$

where ln f(T) denotes the natural logarithm (to base e) of f(T).

If a sum of 1 is invested at an instantaneous rate of interest δ, the amount of interest which will accrue over a period of duration t can be found by integration of (3) from T = 0 to T = t:

$$\begin{aligned}\int_0^t 1.\delta dT &= \int_0^t \frac{d}{dT} \ln f(T)\, dT \\ &= \ln f(t) - \ln f(0) \\ &= \ln \frac{f(t)}{f(0)}\end{aligned}$$

Taking the anti-logarithms by exponentiating both sides to the power e:

$$\exp\left[\int_0^t \delta\, dT\right] = \frac{f(t)}{f(0)}$$

$$\boxed{\therefore f(t) = f(0)\exp\left[\int_0^t \delta dt\right]} \qquad (4)$$

At this point a numerical example will assist to clarify the implications of equation (4).

Consider a person investing R10 in a security about which he knows nothing. After one year the security is redeemed for a sum of R10,50. The average interest rate for the period (equation (1)) would be calculated as

$$R = \frac{10{,}50 - 10{,}00}{10{,}00} \times 100 = 5\% \text{ per annum}$$

Applying the data to equation (4), where $f(0) = R10{,}00$ and $f(1) = R10{,}50$ and $t = 1$,

$$10{,}50 = 10{,}00 \exp\left[\int_0^1 \delta \, dt\right]$$

$$1{,}05 = \exp\left[\int_0^1 \delta \, dt\right]$$

$$= \exp\left[\delta\right]$$

provided δ is time invariant.

Taking natural logarithms on both sides of the equation,

$$\ln 1{,}05 = \delta$$

From tables of natural logarithms,

$$\delta = 0{,}04879$$
$$= 4{,}879\% \text{ per annum.}$$

This example shows that the instantaneous or *continuous rate* of interest is 4,879% per annum and is equivalent to an *effective rate* of 5% per annum. It also shows that interest can be treated as a smooth, continuous phenomenon.

The above statement can be restated mathematically. If i is the effective rate of interest, then after the lapse of one year the magnitude A of an investment P is P (the original principal) plus iP (the interest which has been earned).

i.e. $A_1 = P + iP$
$= P(1+i)$.

After two years have passed, the investment will have grown to

$$A_2 = A_1\,(1+i)$$
$$= P(1+i)\,(1+i)$$
$$= P(1+i)^2.$$

In general terms, after t years,

$$A_t = P(1+i)^t \qquad (5)$$

By application of (4) to this problem, assuming that δ is not a function of time but is constant,

$$f(t) = A_t = P \exp\left[\int_0^t \delta \, dt\right]$$

or $$A_t = Pe^{\delta t} \qquad (6)$$

Equating (5) and (6):

$P(1 + i)^t = Pe^{\delta t}$ and, cancelling P:

$$e^{\delta t} = (1+i)^t \qquad (7)$$
$$\therefore\ e^{\delta t} = (1+i)^t$$

$$\boxed{\delta = \ln(1+i)} \qquad (8)$$

Equation (8) is thus a statement of the relationship between the continuous rate of interest δ and the effective rate of interest i.

(Some authors also refer to δ as being the "force of interest".)

3 THEORY OF DISCOUNTING

The ideas covered in Section 2 were directed at determining the value to which an investment would grow over a given period of time at a given rate of interest. This section looks at the same topic, but "in reverse".

The question being posed is:

> "What sum must be invested now at a given rate of interest so that it might grow to a desired magnitude by the end of a specified period?"

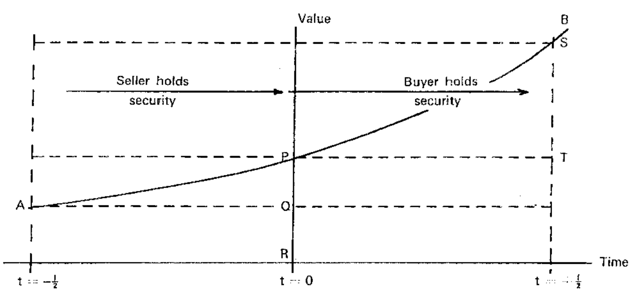
The sum to be invested today is termed the *present value* of the desired future amount. The rate of interest is referred to as the *discounting rate*.

Denoting the present value as p and the future value as A, the following relationship is obvious:

$$pe^{t\delta} = p(1+i)^t = A$$
$$\therefore \quad p = Ae^{-t\delta} = A\,(1+i)^{-t} \qquad (9)$$

Thus it is clear that the interest rates used in the compounding process are also used in determining the discount factors. This fact is very important in the case of fixed interest securities, because it allows the buyer of a security to agree with the seller on the price to be paid. The seller will want to ensure that the proceeds of the transaction are sufficient to give him a return on his investment equal to the yield rate; he will be looking at the question from an interest point of view. The buyer of the security will wish to see that the price which he pays will be such that the future coupons plus maturity repayment of principal will constitute an acceptable rate of return on his outlay; he will be assessing the transaction from a discounting point of view. Provided the two parties can agree on a yield rate, they will both be looking at the same question, albeit from different angles. An agreed price is then possible.

A graphical representation of the transaction is given below. From time $t = -\frac{1}{2}$ to time $t = 0$ the seller of the stock accumulates a return depicted by PQ. From $t = 0$ to $t = +\frac{1}{2}$, the buyer will earn his return which will ultimately be ST. If both parties agree to the yield rate, the curve AB is defined and the price at which the transaction can take place is mutually agreed upon as being PR. The seller will arrive at this price by accumulating interest on his investment whilst the buyer will reach the same answer by discounting the future sums which will be paid to him.



In concluding the introductory concepts, it is important to stress the fact that interest grows *continuously*, and for this reason the exponent "t" in equations (5), (6), (7) and (9) may be non-integer (i.e. fractional).

4 APPLICATION OF DISCOUNTING TO THE VALUATION OF SECURITIES

The purchaser of a fixed-interest bearing security invests his principal with a view to obtaining

- a regular series of a fixed number of fixed coupon payments,
- a lump sum upon maturity corresponding to the nominal value of the paper in question.

The amount that he will be prepared to invest in a specified security will depend on the yield to redemption (also sometimes called yield to maturity) which he wishes to earn on his investment. The investor assumes that all money paid to him by the investment are to be reinvested immediately upon receipt in instruments offering the yield rate. The theory of compound interest and discounting is therefore totally applicable.

4.1 Development of the formula for calculating the PV of an annuity

Annuities may be classified into three categories:

- *Immediate:* First payment is made at the of the first period;
- *Deferred:* First payment is deferred several periods, and payments are made regularly thereafter;
- *Annuity-due:* The first payment is made at the beginning of the first period.

The pricing of fixed interest securities is best treated by considering the coupons to constitute an *immediate* annuity. Such an annuity can be pictorially represented as follows:

```
  c   c   c             c     c     c
|---|---|---|- - - - - -|-----|-----| Time
B   1   2   3          F–2   F–1    F Scale
```

In the diagram are shown F payments of value c paid at the end of each of the F periods. The present value of each payment at time B (see extreme left of diagram) can be found by discounting by means of equation (9):

PV of first payment $= c\,(1+i)^{-1}$

PV of second payment $= c\,(1+i)^{-2}$

PV of third payment $= c\,(1+i)^{-3}$

PV of (F–1) th payment $= c\,(1+i)^{-(F-1)}$

PV of F th payment $= c\,(1+i)^{-F}$

Adding the present values of all the payments gives the present value of the annuity, denoted as PVB.

$$PVB = c(1+i)^{-1} + c(1+i)^{-2} + c(1+i)^{-3}$$

$$+ \ldots + c(1+i)^{-(F-1)} + c(1+i)^{-F}$$

$$= c(1+i)^{-1}\left[(1 + (1+i)^{-1} + (1+i)^{-2}\right.$$

$$\left. + \ldots + c(1+i)^{-(F-2)} + c(1+i)^{-(F-1)}\right]$$

The term in square brackets is a geometric series of F terms and common ratio $(1+i)^{-1}$ and may be summed.

$$\text{i.e. } PVB = c(1+i)^{-1}\left[\frac{1-(1+i)^{-F}}{1-(1+i)^{-1}}\right]$$

$$= c\left[\frac{1-(1+i)^{-F}}{i}\right] \qquad (10)$$

Equation (10) is the general equation for finding the present value of an annuity of F payments of c when the discount rate is i.

The argument may now be extended by adding an additional payment to the stream at B and by calculating a PV at an earlier time A. (This is the usual state of affairs when a stock transaction takes place between coupon dates). The graphical representation of the situation is then as follows:

```
    c     c     c     c           c     c     c
|---|-----|-----|-----|- - - - - -|-----|-----|
A   B     1     2     3          F–2   F–1    F
```

In the above diagram are depicted F full periods plus a fraction of a period (A to B) and F+1 payments of c. The PV at B of all the payments *excluding that made at B* is given by equation (10) above. To that result must be added the additional payment now being made at B. The PV at B of *all* the payments then becomes:

$$PVB' = c + c\left[\frac{1-(1+i)^{-F}}{i}\right]$$

$$= c\left[1 + \frac{1-(1+i)^{-F}}{i}\right]$$

$$= c\left[\frac{i+1-(1+i)^{-F}}{i}\right]$$

$$= c\,(1+i)\left[\frac{1-(1+i)^{-F-1}}{i}\right] \qquad (11)$$

To get the PV of the same set of payments at A, it is necessary to discount the value in (11) over the period B to A. Let the number of days in this period be E and the number of days in a normal interval between payments be J. The extra discounting must therefore be done for one E/Jth of a period. Denote this fraction as θ. The PV of all the payments at A can then be written as

$$PVA = \frac{c(1+i)}{(1+i)^{\theta}}\left[\frac{1-(1+i)^{-F-1}}{i}\right]$$

$$= c(1+i)^{1-\theta}\left[\frac{1-(1+i)^{-F-1}}{i}\right]$$

Multiply both numerator and denominator by

$$(1+i)^{F+\theta}:$$

$$PVA\ \frac{c(1+i)^{1+F}}{(1+i)^{F+\theta}}\left[\frac{1-(1+i)^{-F-1}}{i}\right]$$

$$= \frac{c}{(1+i)^{F+\theta}}\left[\frac{(1+i)^{F+1}-1}{i}\right]$$

In order to simplify the expression, make the substitution

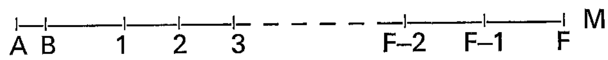
$$G = F + \theta$$
$$= F + E/J$$

(i.e. G = total number of full periods remaining *plus* the additional fraction of a period. G is therefore the total length of time over which discounting takes place.)

$$PVA = \frac{c\left[(1+1)^{F+1} - 1\right]}{i.(1+i)^G} \quad (12)$$

4.2 Present value of a lump sum

Consider the same time span as was used in Section 4.1 in the development of equation (12). A lump sum payment M will be made at the end of the period, and the purpose of this exercise is to find its present value at A.



As before, the number of full periods is F and the total time is F + θ = G periods. Using the justification for discounting over fractional periods as developed in Section 3 and using equation (9), the present value at A of M can be written as

$$PVA_M = M(1+i)^{-G} \quad (13)$$

4.3 Valuation of fixed interest securities having bi-annual coupon payments

At this stage, only one point remains to be clarified before the aforegoing theory can be applied to the valuation of securities carrying bi-annual coupons: interest nomenclature.

It is customary, in South Africa if not elsewhere in the world, to quote fixed interest securities at their *nominal yield* rates. A nominal annual rate of interest is merely the ratio of interest paid per year to capital invested, *assuming no reinvestment of the interest until the end of the year.* For example, a nominal rate of 10% payable twice annually means that two payments of 5% spaced six months apart, are made.

If the interest payments were immediately capable of reinvestment at the same rate of (nominal) 10%, the effective rate would be

$$(1 + (\tfrac{1}{2} \times 0{,}10))^2 = 10{,}25\%.$$

(In the case of bi-annual payments, the effective rate is higher than the nominal rate by the square of half the nominal rate, e.g.

$$(\tfrac{1}{2} \times 0{,}10)^2 = (0{,}05)^2 = 0{,}25\%).$$

Thus, in the case of most, if not all, South African fixed interest securities, the effective yield rate is greater than most investors realise.

Because South African fixed interest securities are quoted at nominal yield rates and have bi-annual coupons, the valuation formula is more easily stated in terms of six-month periods (instead of using the year as the basic unit of time). The effective yield rate for a six month period (used for discounting) is half the nominal annual rate. For example, a stock yielding a nominal 10% p.a. would have an effective rate of 5% per six months.

A person investing in a fixed interest security purchases

(i) A terminal capital redemption amount,

(ii) A series of six-monthly coupons C/2,

which he discounts at a rate Y/2

(i.e. coupon rate is C% per annum and yield to redemption is Y% per annum (nominal). The price of the stock can be obtained by finding the PV of the terminal capital redemption and adding it to the PV of the annuity of coupons. This can easily be done by using equations (12) and (13) with the substitution of Y/200 for i and C/200 for c (where Y and C are per cent):

$$\text{PV of terminal capital redemption} = \frac{M}{(1+Y/200)^G}$$

PV of annuity of coupons =

$$\frac{C/200\left[(1+Y/200)^{F+1} - 1\right]}{Y/200\,(1+Y/200)^G} \times 100$$

(the "100" being the nominal value of the security on which the coupon amount is based). Thus the value, or price of a fixed interest security, is the sum of these two components:

$$\text{Price} = \frac{M}{(1+Y/200)^G} + \frac{C/200\left[(1+Y/200)^{F+1} - 1\right]}{Y/200\,(1+Y/200)^G} \times 100 \quad (14)$$

5 OF CLEAN PRICES AND ACCRUED INTEREST

The price which an investor is prepared to pay for a particular stock having coupon C and period G (in which are F full periods) to run so as to realise a yield to redemption Y, was given in equation (14). However, for tax and accounting reasons, some investors wish to subdivide the price into two components

– Clean (or naked) price (CNP)

– Accrued coupon interest.

The accrued coupon interest is that portion of the next coupon to which the seller is entitled. It is calculated, according to tradition, on a linear basis (pro rata to the length of time that the seller held the stock) at half the annual coupon rate. Strictly speaking, this is incorrect and, as was shown in equation (4), the instantaneous rate should be used. However, because the total price is not affected by the split between CNP and accrued interest, the imperfection will be ignored.

The fraction of the coupon attributable to the seller of the stock is I_s, such that

$$I_s = C/2 \times \frac{\text{number of days elapsed}}{\text{number of days in current coupon period}}$$

The number of days which have elapsed may be written as (J—E). Thus

$$I_s = C/2 \times \frac{J-E}{J}$$

$$= C/2 \times (1 - \frac{E}{J})$$

Now, because $G = F + \frac{}{J}$ it is possible to make the substitution

$$-\frac{E}{J} = F - G.$$

$$\boxed{\therefore\ I_s = C/2 \times (1 + F - G)} \qquad (15)$$

Combination of equation (14) with equation (15) gives the formula for the clean price of a stock:

$$\boxed{CNP = \frac{M}{(1+Y/200)^{G}} + \frac{C/200\,[(1+Y/200)^{F+1} - 1]}{Y/200\,(1+Y/200)^{G}} \times 100 - C/2(1+F-G)} \qquad (16)$$

Sadly, it is true that many dealers in fixed interest securities do not realise that the removal of interest from the price of a stock is calculated according to equation (15). They accept the CNP produced by a financial calculator and then add accrued interest calculated as

$$I'_s = C \times \frac{J-E}{365} \qquad (17)$$

in order to arrive at the selling price. The only correct interest which should be added back to the CNP in order to obtain the full price is that which was removed from it in the first place. In other words, equation (15) must be used for accrued interest calculations.

(Alternatively, the CNP equation must be modified so that the CNP is the difference between equations (14) and (17). However, this is not practicable because the most widely used financial calculator has been programmed with equation (16). Moreover, equation (16) is the least incorrect.)

To conclude, the price of a stock may be pictorially represented as follows:

Clean price Equation (16)	Accrued interest Equation (15)
← Total price or value →	
Equation (14)	

Note that the underlying equation is (14) which is subdivided by equations (16) and (15).

5.1 The clean price of a par stock

It is widely believed that the clean price of a par stock (i.e. a security having equal coupon and yield rates) should always be R100,00%. This is incorrect – a par stock having a terminal maturity redemption value M of R100,00 will have a clean price of R100,00% only on coupon dates. This can be established by examination of equation (16):

$$CNP = \frac{M}{(1+Y/200)^{G}} + \frac{C/200\,[(1+Y/200)^{F+1} - 1]}{Y/200\,(1+Y/200)^{G}} \times 100 - C/2(1+F-G) \qquad (16)$$

If the yield and coupon rates are equal, the substitution of Y for C is possible. For the purpose of this sub-section, M = 100. Equation (16), incorporating the modifications, can therefore be written as:

$$CNP_{par} = \frac{100}{(1+Y/200)^{G}} + \frac{Y/200\,[(1+Y/200)^{F+1} - 1]}{Y/200\,(1+Y/200)^{G}} \times 100 - Y/2(1+F-G)$$

$$= \frac{100(1+Y/200)^{F+1}}{(1+Y/200)^{G}} - Y/2(1+F-G) \qquad (16^1)$$

In the general case where G is not an integer (i.e. when the valuation is performed on any day except a coupon date), equation (16^1) does *not* have the value of R100,00%.

In the special case where the security is valued on a coupon date,

$$F - G = -1$$

or $$G = F+1.$$

Making this substitution into (16^1) causes the second term to become zero and the first is 100. In other words, a par fixed-interest security has a price of R100,00% on its coupon dates *only*.

5.2 The total selling price of a par stock

Equation (14) gave the selling price of a fixed interest security as:

$$\text{Price} = \frac{M}{(1+Y/200)^{G}} + \frac{C/200\left[(1+Y/200)^{F+1} - 1\right]}{Y/200\,(1+Y/200)^{G}} \times 100 \qquad (14)$$

If M = R100,00 and Y = C (i.e. stock is trading at par), the equation may be rewritten as

$$\text{Price} = \frac{100}{(1+Y/200)^{G}} + \frac{Y/200\left[(1+Y/200)^{F+1} - 1\right]}{Y/200\,(1+Y/200)^{G}} \times 100$$

$$= 100\,\frac{(1+Y/200)^{F+1}}{(1+Y/200)^{G}}$$

$$= 100\,(1+Y/200)^{1+F-G}$$

Because the exponent (1+F—G) is a positive number representing the fraction of the current coupon period which has lapsed, the total price of a par stock is always greater than R100,00% between coupon dates.

6 CONCLUSION

If a fixed-interest security is represented as a combination of

- an annuity of coupon payments,
- a lump sum maturity redemption amount

it may be valued by discounting the components at the yield rate. The resultant value (or price) may then be divided between the clean naked price and accrued coupon interest. The application of this method implies that the clean naked price of par stocks is only R100,00% on coupon dates.

It is important to note that the calculation method for accrued coupon interest must be consistent with the formula for the clean naked price.

Appendix: List of symbols used

A_n	—	the value of an investment after compound interest has accrued for n compounding periods
C	—	coupon rate (per cent, not decimal) or amount of coupon payment
CNP	—	clean naked price
c	—	annuity payment amount
E	—	number of days before next coupon
e	—	base of natural logarithms (2,71828)
exp	—	raise to the power e
F	—	number of whole periods
f	—	hypothetical investment
G	—	total time (in periods) to maturity, equal to $F + \theta = F + E/J$
g	—	fractional growth in hypothetical investment
h	—	period over which hypothetical investment is held
I_s	—	proportion of next coupon due to seller
i	—	interest rate
J	—	number of days in compounding period
ln	—	natural logarithm (to base e)
M	—	redemption or maturity amount
P	—	principal invested
PVA	—	present value at A of stream of payments
PVB	—	present value at B of stream of payments
p	—	present value (general term)
R	—	time rate of growth of hypothetical investment
T	—	time (used as argument to f)
t	—	time
Y	—	yield rate (per cent, not decimal)
δ	—	instantaneous or continuous interest rate
θ	—	fraction of period remaining before next coupon