By S. T. Goddard, S. A. Michener and J. A. Rickard*

# An illustrative method for valuing annuities

*Graduate School of Business Administration and Department of Mathematics, University of Melbourne

## 1 INTRODUCTION

In many situations, individuals or institutions wish to know an equivalent dated value of a series of cash flows. If the cash flows are of equal value and occur at equal intervals of time, they are said to take the form of an *annuity*.

The *future value* of an annuity is the sum of all the cash flows and the compound interest on them accumulated from the beginning of the first payment interval to the end of the last payment interval of the annuity.

In many situations, a cash flow occurs at the termination of an annuity which is not part of the series of regular payments. Such a cash flow is known as a *balloon payment* and must be considered when finding the present value of the annuity.

The *present value* of an annuity is the sum of the present values of all the cash flows of an annuity and the present value of the balloon payment.

Although the values of an annuity can be readily determined by tables and modern calculators, the specific answer obtained provides no measure of the sensitivity of the result to changes in the variables. In Section 4 a graphical method for calculating the value of an annuity is presented in the form of a nomogram. The nomogram that has been constructed is easy to use (in fact the user only requires a ruler and pencil) and clearly illustrates the changes in the value of the annuity when it is subject to changes in the values of the input parameters. An example of the benefits of using the nomogram for this type of sensitivity analysis can be seen in Section 5.

## 2 TYPES OF ANNUITIES

In this paper, expressions will be derived for the valuation of the four main types of annuities that occur in practice.

An *ordinary annuity* arises when the cash flows occur at the end of the periods, such as, for example, the repayments on a loan or mortgage. If the payments are made at the beginning of the periods, the annuity is known as an *annuity due*. This type of annuity often occurs in a rent or lease agreement.

The next two types of annuities are both extensions of the above definitions and depend on whether the original annuity is an ordinary annuity or an annuity due. If the payments of an annuity do not commence until a number of compounding periods into the future, the annuity is known as a *deferred annuity*. For example, when purchasing a car, the buyer may be allowed a certain period of "free" use before repayments begin. If the series of equal cash flows continues forever, the annuity is known as a *perpetuity*. An example of a perpetuity is the dividends paid on non-redeemable shares.

## 3 CALCULATING THE PRESENT AND FUTURE VALUES OF ANNUITIES

In general, for simplification, an annuity is constructed so that the compounding of interest occurs on the same date as the annuity payments. The analysis presented in this section will be subject to this assumption, however the assumption will be relaxed in Section 6.

To define the variables of this problem, let

$n$ = total number of payments
$i$ = nominal interest rate (per compounding period)
PMT = periodic payment
BAL = balloon payment

### Future values

For an ordinary annuity, the payments are made at the end of the periods, so the final payment will have a future value of PMT. The second last payment will be compounded for one period, giving a value of PMT$(1+i)$. This argument follows for all other payments, thus giving the future value of the first payment as PMT$(1+i)^{n-1}$. Hence, the future value of the ordinary annuity, $FV_o$, is

$$FV_o = PMT\left[1+(1+i)+\ldots+(1+i)^{n-1}\right] \qquad (1)$$

Since (1) is a geometric progression with initial term PMT and common ratio $(1+i)$,

$$FV_o = PMT\left[\frac{(1+i)^n-1}{i}\right] \qquad (2)$$

A similar method can be used to derive the future value of the annuity due, remembering that the payments are made at the beginning of the periods, so an "extra" period of interest applies to each term. Thus, the future value of the annuity due, $FV_d$, is

$$FV_d = PMT\left[(1+i)+(1+i)^2+\ldots+(1+i)^n\right] \qquad (3)$$

which reduces to

$$\left.\begin{aligned} FV_d &= PMT\,(1+i)\left[\frac{(1+i)^n-1}{i}\right] \\ &= (1+i)FV_o. \end{aligned}\right\} \qquad (4)$$

By definition of the future value of an annuity, the future value of a deferred annuity is calculated in the same way as the future value of an ordinary annuity or an annuity due. However, in the case of a deferred annuity, the amount of the future value is not available until the term of the deferred annuity is finished, which is longer than the term of an ordinary annuity or an annuity due by the period of deferment.

As previously defined, the future value of an annuity is the sum of all the cash flows and the compound interest accumulated on them to the end of the term of annuity. For a perpetuity, the payments continue forever thus making the future value of the perpetuity indeterminate.

### Present values

To determine the present value of an annuity, the present values of both the regular payments and the balloon payment must be considered. Although the addition of the balloon payment term complicates the expressions further, the applications of the expressions are considerably extended. For example, the balloon payment term may refer to a final extra payment that is used to pay off a loan completely, or to the selling of a piece of leasing equipment that has earned regular payments as rent. As stated in Section 1, the balloon payment is assumed to be made at the termination of the annuity; that is, at the end of period n for each type of annuity.

Since the present value of an annuity is the sum of the present values of all the regular payments and the present value of the balloon payment, the present value of an ordinary annuity, $PV_o$, is

$$PV_o = (1+i)^{-n}\left\{PMT\left[1+(1+i)+\ldots+(1+i)^{n-1}\right]\right\} + BAL(1+i)^{-n}$$
$$= PMT\left[\frac{1-(1+i)^{-n}}{i}\right] + BAL(1+i)^{-n}. \qquad (5)$$

Similarly, the present value of an annuity due, $PV_d$, is

$$PV_d = (1+i)^{-n}\left\{PMT\left[(1+i)+(1+i)^2+\ldots+(1+i)^n\right]\right\} + BAL(1+i)^{-n}$$
$$= PMT(1+i)\left[\frac{1-(1+i)^{-n}}{i}\right] + BAL(1+i)^{-n}.$$
$$= (1+i)PV_o - iBAL(1+i)^{-n}. \qquad (6)$$

The expression for the present value of a deferred annuity can be found by assuming that there are n+m equal payments to be made with the first m payments withheld. This would then be equivalent to making n payments beginning at a time of m periods into the future.

For an ordinary annuity, the present value of the deferred ordinary annuity, $PVD_o$, can be found by assuming that all n+m payments were made and then subtracting the present value of the first m payments from the present value of all the payments and the present value of the balloon payment. Thus

$$PVD_o = PMT\left[\frac{1-(1+i)^{-n-m}}{i}\right] + BAL(1+i)^{-n-m} - PMT\left[\frac{1-(1+i)^{-m}}{i}\right]$$
$$= PMT\left[\frac{(1+i)^{-m}\{1-(1+i)^{-n}\}}{i}\right] + BAL(1+i)^{-n-m}. \qquad (7)$$

A similar method can be used to find the present value of the defered annuity due, $PVD_d$, giving

$$PVD_d = PMT(1+i)\left[\frac{1-(1+i)^{-n-m}}{i}\right] + BAL(1+i)^{-n-m} - PMT(1+i)\left[\frac{1-(1+i)^{-m}}{i}\right]$$
$$= PMT\left[\frac{(1+i)^{-m+1}\{1-(1+i)^{-n}\}}{i}\right] + BAL(1+i)^{-n-m}$$
$$= (1+i)PVD_o - i\,BAL(1+i)^{-n-m}. \qquad (8)$$

Since the payments of a perpetuity continue forever, the expression for the present value of a perpetuity may be found by letting n tend to infinity and determining the limiting value of the present value of the annuity. Thus, for the present value of an ordinary perpetuity, $PVP_o$, the expression is

$$PVP_o = \lim_{n\to\infty}\left\{PMT\left[\frac{1-(1+i)^{-n}}{i}\right]\right\} = \frac{PMT}{i} \qquad (9)$$

since $\lim_{n\to\infty}\frac{1}{(1+i)^n} = 0$ for all non-negative i.

Similarly, the present value of a perpetuity due, $PVP_d$, is

$$PVP_d = \lim_{n\to\infty}\left\{PMT(1+i)\left[\frac{1-(1+i)^{-n}}{i}\right]\right\}$$
$$= PMT\left(\frac{1+i}{i}\right)$$
$$= (1+i)PVP_o. \qquad (10)$$

When considering perpetuities, no balloon payment need be taken into account since the payments continue forever, which implies that no final payment exists.

## 4 THE NOMOGRAM

As previously mentioned, a graphical method of calculation of an equivalent dated value of an annuity in the form of a nomogram is presented. A nomogram is a collection of graphs that share common axes. Nomograms have found recent applications in a number of areas, including the valuation of options and warrants [1] and the comparison of flat, nominal and effective rates of interest [2].

### Future values

Firstly, a nomogram will be constructed so as to give future values of all the annuities. The nomogram is constructed using equations (2) and (4).

Initially, a set of contours n = constant are used to plot i against

$$\frac{(1+i)^n-1}{i}.$$

After these curves are drawn, a set of contours PMT = constant may be used to plot

$$\frac{(1+i)^n-1}{i}.$$

against $FV_o$. The value of $FV_o$ may then be found from the $FV_o$-axis by using the suitable contour PMT = constant. Plotting $FV_o$ against $FV_d$ with contours i = constant then allows the value of $FD_d$ to be found from the $FV_d$-axis.

Since the equations for the future values of the annuities are quite simple, the fourth quadrant of this nomogram is not required (see Figure 1).

*Example* (see Figure 1)

"Find the future values of the annuities with regular payments of $100 every 6 months for 15 years, if the nominal interest rate is 4% per 6 months."

Firstly read off the nominal interest rate of 4% on the i – axis

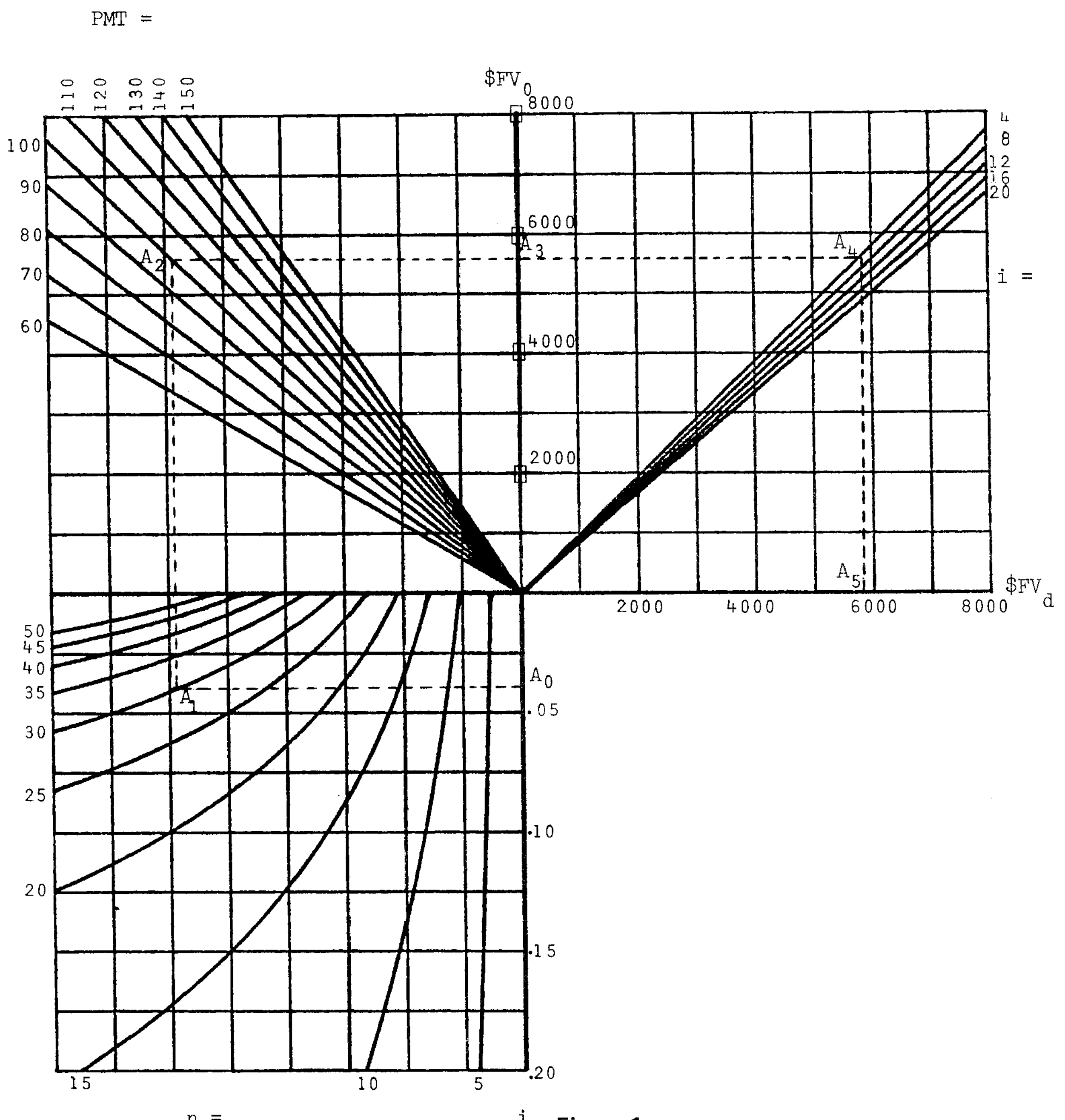


**Figure 1**
Nomogram for calculating the future value of ordinary annuities and annuities due.

at $A_0$ and draw a horizontal line to the left until it intersects the n = 30 contour at $A_1$. Next, draw a vertical line at $A_1$ until this line intersects the PMT = \$100 contour at $A_2$. A horizontal line is constructed at $A_2$ and drawn until it intersects the $FV_o$ – axis at $A_3$ which gives the value of $FV_o$. The last horizontal line, $A_2A_3$, may be continued until it meets the i = 4% contour at $A_4$. The vertical line constructed at $A_4$ intersects the $FV_d$ – axis at $A_5$ which gives the value of $FV_d$. This solves the future value annuity problem, giving

$FV_0$ = \$5 610.00 and $FV_d$ = \$5 840.00,

which yields errors of less than 1% in each case when compared to the actual values $FV_o$ = \$5 608.49 and $FV_d$ = \$5 832.83.

As previously mentioned, the nomogram also gives the approximate future values of the deferred annuities

$FVD_o$ = \$5 610.00 and $FVD_d$ = \$5 840.00,

but the amounts are not available until the term of the deferred annuity is finished.

**Present values**

As might be expected from the more complicated formulae, the nomograms for the present values of all the annuities are more complex, and each type of annuity requires its own nomogram.

For both the ordinary annuity and the annuity due, a similar method as described above is used to construct a nomogram (see Figures 2 and 3), except that five sectors are required in each case. There is, however, one large difference in the use of these nomograms. Apart from being able to find the present value of only one type of annuity per nomogram, the value is obtained from the contours in the first sector rather than being read off an axis. Although this may cause the need for some interpolation, the number of curves in the first sector should make any errors very slight.

*Example*

"Find the present value of the annuities with regular payments of \$100 every 6 months for 15 years if the nominal interest rate is 4% per 6 months, and the balloon payment is \$500."

Firstly, consider the case of an ordinary annuity (see Figure 2). Read off the nominal interest rate of 4% on the i – axis at $B_0$ and draw a line perpendicular to the axis until it meets

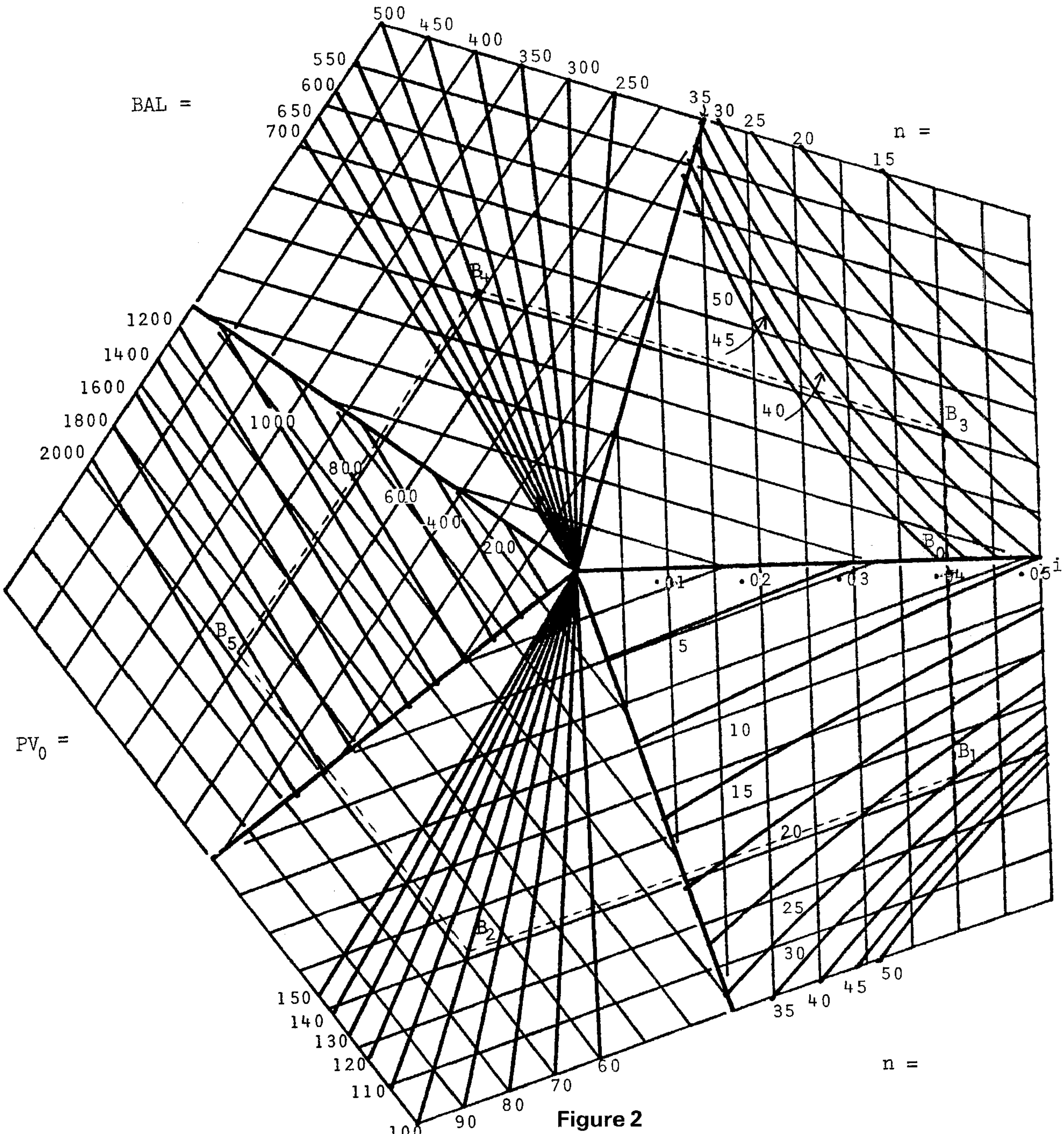


**Figure 2**

Nomogram for calculating the present value of ordinary annuities.

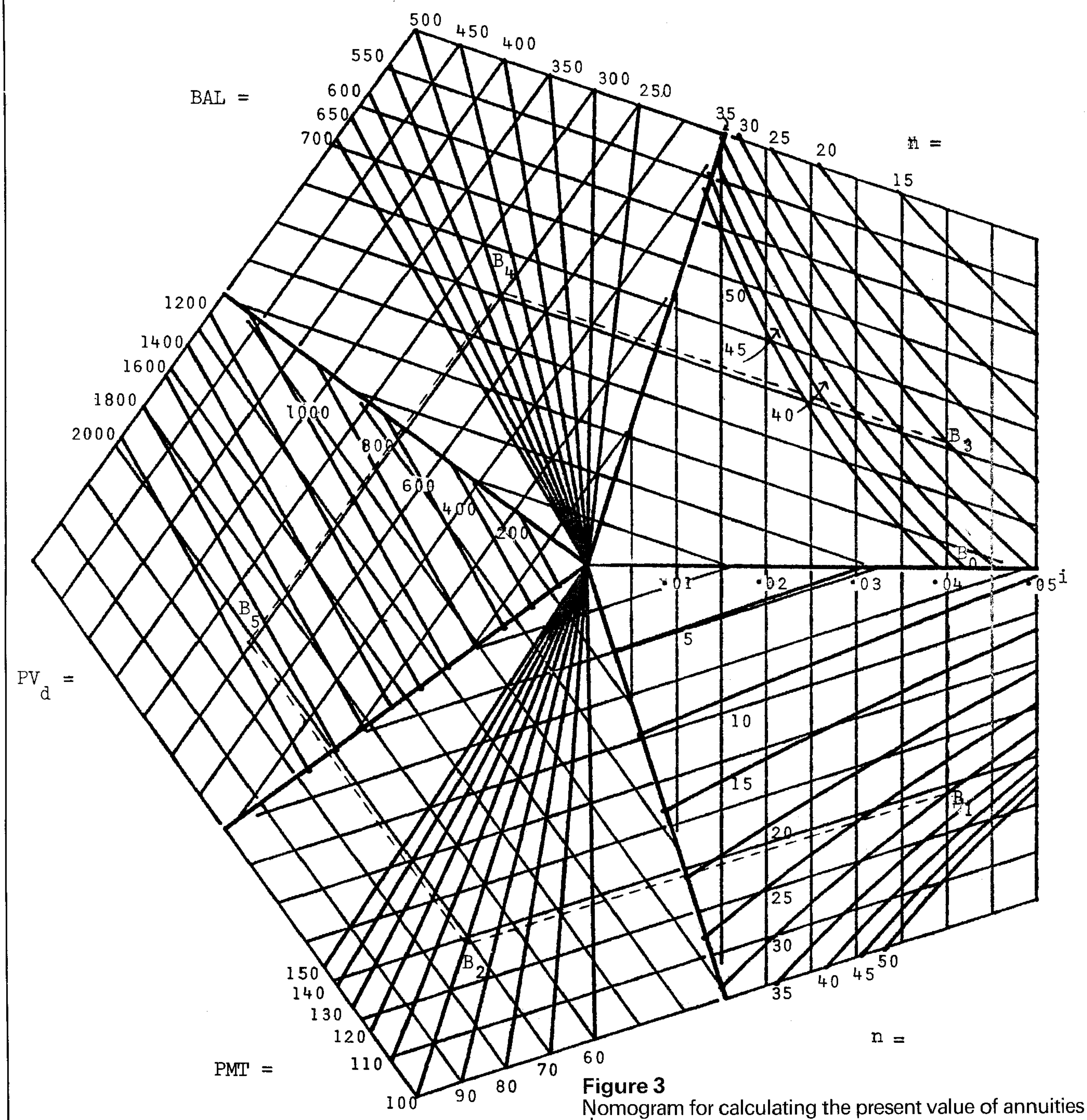


**Figure 3**
Nomogram for calculating the present value of annuities due.

the n = 30 contour at $B_1$, then draw a line parallel to the grid from $B_1$ to the PMT = \$100 contour at $B_2$. At $B_2$, draw a line parallel to the grid which passes through all the contours in the next sector.

Next, returning to $B_o$, draw a line perpendicular to the i – axis until it meets the contour at $B_3$. Draw a line parallel to the grid from $B_3$ to the BAL = \$500 contour at $B_4$. Construct a line from $B_4$ parallel to the grid until this line intersects the line from $B_2$ at $B_5$. Using the $PV_o$ = constant contours and the point of intersection, the value of $PV_o$ = \$1 880 may be interpolated, which yields an error of less than 1% when compared to the correct answer of $PV_o$ = \$1 883.36.

The solution to this example for an annuity due is shown in Figure 3 where the value $PV_d$ = \$1 950 is obtained, yielding an error of less than 1% since the correct answer is $PV_d$ = \$1 952.53. These two monograms thus solve the present value problem for these two types of annuities.

Although no nomograms are presented for calculating the present values of deferred annuities, the values can be found indirectly from Figures 2 and 3 by using an annuity of length n+m instead of n and then subtracting

$$PMT\left[\frac{1-(1+i)^{-m}}{i}\right]$$

for the ordinary deferred annuity and

$$PMT(1+i)\left[\frac{1-(1+i)^{-m}}{i}\right]$$

for the deferred annuity due. Note that the two values to be subtracted can also be found from Figures 2 and 3, using a term of annuity of length m.

The nomograms used in the above example were of normal desk pad size (45 cm x 65 cm), which explains the accuracy obtained.

## 5 SENSITIVITY ANALYSIS

In this section, the benefits of the nomogram for sensitivity analysis will be illustrated by changing the values of the variables of the problem. By choosing one variable and changing its value by certain amounts, the nomogram can be used to measure the sensitivity of the annuity to changes in that particular variable.

For example, Figure 2 shows the changes in the present value of an ordinary annuity when the regular payments are increased and decreased by \$10, yielding present values of \$2 050.00 and \$1 550.00 respectively (with errors of less than 1%).

Another interesting application of sensitivity analysis is to consider the effects of changing the interest rate. In Figure 3, some valuations are shown for the present value of an annuity due where the interest rate is varied while keeping the regular payments and balloon payment constant. The present value for the above example (with $i = .04$) is shown, as are the present values for the example with $i = .03$ and $i = .05$, yielding present values of \$2 240.00 and \$1 710.00 respectively (with errors of less than 1%).

This type of analysis can be very useful when deciding the value of regular and balloon payments with respect to the changes in the present and future values of the annuity.

## 6 GENERAL ANNUITIES

As previously mentioned, some restrictions have been placed on the problem so far with regard to the dates of compounding of interest and the annuity payments. For the case of a general ordinary annuity, let

GPMT = number of annuity payments
s = number of compounding periods per annum
t = number of general annuity payments per annum
i = nominal interest rate per compounding period

It is possible to replace this general ordinary annuity with an equivalent ordinary annuity with payments made at the end of the compounding periods, providing that the interest rates of the two annuities are equivalent and that their values on any date are the same. Let PMT be the ordinary annuity payment which replaces GPMT and j the nominal interest rate per general annuity payment period that is equivalent to i per compounding period.

The relationship between i and j is given by

$$(i+j)^t = (1+i)^s \qquad (11)$$

which may be solved explicitly for j to give

$$j = (1+i)^{s/t} - 1. \qquad (12)$$

Since the values of the annuities are equal on any date,

$$PMT\left[\frac{(1+i)^s-1}{i}\right] = GPMT\left[\frac{(1+j)^t-1}{j}\right] \qquad (13)$$

However, from the equivalence of the interest rates in (11), (13) reduces to

$$\frac{PMT}{i} = \frac{GPMT}{j}$$

Using (12), the payment of the equivalent ordinary annuity is

$$PMT = i\,GPMT\left[(1+i)^{s/t}-1\right]^{-1} \qquad (14)$$

from which the nomograms in Figures 1 and 2 may be used to calculate the equivalent future value and present value of the general ordinary annuity.

The case of a general annuity due may be treated in an analogous manner. Assuming the variables as defined above, the general annuity due can be transformed into a general ordinary annuity by transferring each GPMT to the end of each year. This means that equation (14) could then be used, but the values of the general annuity due would exceed the value of the ordinary general annuity by

$$GPMT(1+i)^s - GPMT \qquad (15)$$

over each one year period. However,

$$GPMT(1+i)^s - GPMT = i\,GMPT\left[\frac{(1+i)^s-1}{i}\right] \qquad (16)$$

which is the future value of an ordinary annuity with regular payments i GPMT per interest period for one year. Since the general annuity payments GPMT are replaced by equivalent payments PMT made per compounding period, the payments PMT are both i GPMT more than they would be if the general annuity were an ordinary one. Thus

$$PMT = i\,GPMT\left[(1+i)^{s/t}-1\right]^{-1} + GPMT\,i$$

$$= i\,GPMT\left[1-(1+i)^{s/t}\right]^{-1} \qquad (17)$$

Again, equation (17) may be used in Figures 1 and 3 to calculate the future value and present value of the general annuity due.

It is also possible to generalise the results obtained by considering continuous compounding of interest, rather than the cases where interest is added to principal at the end of discrete periods of time. This is equivalent to interest compounded s times per year, where s tends to infinity. That is,

$$\lim_{S\to\infty}\left(1+\frac{i}{s}\right)^s = e^i. \qquad (18)$$

Thus, the effective rate of interest is

$$r = e^i - 1. \qquad (19)$$

Since this case is not in general use, it is only noted here and not considered in detail.

### References

[1] E. Dimson, "Instant Option Valuation", paper presented at the conference on "Investing in Options", London, November 1976.

[2] J. A. Rickard, "A nomogram relating flat, nominal and effective rates of interest", *Accountancy,* (1979), to appear.