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The true cost of loans with rests between adjustment of principal

In a rested interest loan, the frequency of adjustment of the outstanding principal is different to the repayment frequency. In this paper, the structure of rested interest loans is investigated and compared to more conventional loans in which the frequencies are the same.

A nomogram for the determination of the regular repayments associated with a loan is presented. In addition, the true cost of the loan is obtained using both the effective rate of interest and the equivalent reducing rate of interest per adjustment period. The sensitivity of these rates with respect to the number of repayments per adjustment period is also discussed.

1. Introduction

It is a not uncommon practice amongst a variety of financial institutions to adjust the principal outstanding on loans at a frequency which differs from the loan repayment frequency. Indeed, one often encounters loan advertisements which indicate the difference in these frequencies, for example, "... monthly repayments and six-monthly adjustments of principal ...". A number of Australian banks employ this procedure with the home loans which they offer, although this is *not* always made clear to the borrower. Whilst the costs or benefits, depending on whether one is a borrower or lender, associated with different repayment frequencies have recently been analysed by Rickard and Russell[8], we are not aware of any attempt to determine the true cost (benefit) associated with these "rested interest" loans.

In this paper, we examine the structure of "rested interest" loans, indicating clearly their relation to and, in general, difference from the more common and better understood loans in which the principal is adjusted at the time of each (regular) repayment. Of course, the latter is, in fact, a special case of a loan with rested interest, namely that in which the "rest" disappears. We present a nomogram which determines the amount of the (regular) repayment instalment for a rested interest loan, given the loan duration, the repayment and adjustment frequencies, the (reducing) interest per adjustment period and the amount borrowed. The nomogram is extremely simple to use and requires only a pencil and ruler for its operation; the user is not required to do any calculations whatsoever. A variety of nomograms have recently been devised to assist in the solution of a range of financial problems, see, for example, Dimson[1], Goddard, Michener and Rickard[2,3], Rickard[4,5,6], Rickard and Russell[7] and Stanton and Rickard[9].

In the paper, we determine precisely the true cost of a "rested interest" loan, using for illustration both the *effective rate of interest* applicable to the loan and the *equivalent reducing rate of interest per repayment period.* The latter corresponds to the reducing rate of interest applicable to an identical loan (the same loan duration, repayment frequency, loan amount and repayment instalment) with *simultaneous* repayments and adjustment of principal.

We illustrate and discuss how the equivalent nominal rate and the effective rate vary with the *number of repayments per adjustment period.*

2. The general rested interest loan

We wish to determine an expression for the amount of the regular repayment instalment, $R, required to service (amortise) a loan of $S over y years given that adjustment of principal occurs n times per annum and that m repayments are required per adjustment period. We suppose that the nominal rate of interest on the loan, 100 i% per annum, is given. Hence, the (reducing) rate of interest per *adjustment* period is $100\frac{i}{n}$%. Denote by S_j the amount outstanding on the loan at the *beginning* of the jth repayment period. Hence, prior to the first adjustment of principal, that is, for $j < m + 1$, we have

$$S_1 = S$$
$$S_j = S - (j - 1)R. \qquad (2.1)$$

At the end of the mth repayment period, after m instalments of $R have been repaid, the principal outstanding is adjusted for the first time. This is achieved by adding the total accrued interest to S_m. The total interest charge accrued is the sum of the interest accrued during the first m repayment periods. The interest accrued in period j ($1 \leq j \leq m$), I_j, is given by

$$I_j = \frac{i}{mn} S_j. \qquad (2.2)$$

From equations (2.1) and (2.2) it follows that

$$S_{m+1} = S_m - R + \sum_{j=1}^{m} I_j, \qquad (2.3)$$

which, on summing the arithmetic series in (2.3), reduces to

$$S_{m+1} = (1 + \frac{i}{n})S - \frac{1}{2}R(2m + (m-1)\frac{i}{n}). \qquad (2.4)$$

Note that (2.4) reduces to the familiar result

$$S_{m+1} = (1 + \frac{i}{n})S - R \text{ when } m = 1,$$

that is, for simultaneous repayment and adjustment.

Proceeding as above, it is straightforward to obtain an expression for the principal outstanding after the pth adjustment ($0 \leq p \leq ny$). This is given by

$$S_{pm+1} = S(1 + \frac{i}{n})^p - \frac{1}{2}R(2m + (m-1)\frac{i}{n})\sum_{j=0}^{p-1}(1 + \frac{i}{n})^j \qquad (2.5)$$

and, once again, the familiar result for simultaneous repayment and adjustment follows immediately on putting m = 1. Since the loan is to be completely repaid after the (ny)th adjustment, that is, after mny repayment instalments of $R, it follows that $S_{mny+1} = 0$ and hence, from (2.5), that

$$R = \frac{2\frac{i}{n}S(1 + \frac{i}{n})^{ny}}{[2m + (m-1)\frac{i}{n}][(1 + \frac{i}{n})^{ny} - 1]}. \qquad (2.6)$$

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Again, we note that the result for the familiar periodic and simultaneous repayment and adjustment loan is recoverable from (2.6) on putting $m = 1$. This gives

$$R^* = S\frac{i}{n} - A^{-1}_{\overline{N}|i} = S\frac{i}{n}[1 - (1 + \frac{i}{n})^{-N}]^{-1}, \qquad (2.7)$$

where $N = ny$ and $A^{-1}_{\overline{N}|i}$ is the *capital recovery factor* or *annuity factor.*

3. Nomogram for determination of repayment instalment

Using the structure of equation (2.6), it is possible to devise a nomogram suitable for determining R given m, N, S and $\frac{i}{n} = i^*$, say. Note that n represents the total number of adjustment periods while i^* is the reducing interest rate per adjustment period.

It will be convenient to introduce the variables u and v defined by

$$u = \frac{i}{n}A^{-1}_{\overline{N}|i} = i^* - [1 - (1 + i^*)^{-N}]^{-1}, \qquad (3.1)$$

$$v = m + \frac{1}{2}(m - 1)i^*, \qquad (3.2)$$

from which it immediately follows that

$$R = \frac{uS}{v}. \qquad (3.3)$$

It is clear from (3.1) that we may plot u as a function of i^* for given values of N; this results in a series of contours N = constant in the u, i^* plane. Similarly, from (3.2), we may plot v against i^* for given values of m. In this case, the contours m = constant are straight lines. Finally, we note that plotting uS against u for constant S and v against uS for constant R yields straight line contours S = constant and R = constant, respectively.

The nomogram of Figure 1 was constructed using equations (3.1)–(3.3) and the properties discussed above. To illustrate how to use the nomogram, suppose we are given $i^* = i^*_0$, $N = N_0$, $S = S_0$, $m = m_0$ and wish to find $R = R_0$, say. To use the nomogram the user begins, for example, at P_0 on the i^*-axis (the left-hand horizontal axis) with $i^* = i^*_0$. The user next constructs the vertical line $P_0 P_1$ to meet the $N = N_0$ contour in the third quadrant at P_1. The horizontal line $P_1 P_2 P_3$ is then constructed to meet the $S = S_0$ contour at P_3 in the fourth quadrant. The intersection of $P_1 P_2 P_3$ with the u-axis occurs at P_2 and determines $u = u_0$, say, but this is of no real interest to us. The user now constructs the upward vertical line $P_3 P_4$ and extends this into the first quadrant.

The intersection of this line with the uS axis gives the repayment amount $u_0 S_0$ required to amortise the loan using Nm *simultaneous* repayments and adjustments. In fact, for this case ($m = 1$), the problem is completely solved using only the third and fourth quadrants; for further details see Rickard.[5]

To continue, the user now returns to P_0 on the i^*-axis and extends $P_1 P_0$ upwards to meet the $m = m_0$ contour at P_5 in the second quadrant. The horizontal line $P_5 P_6$, extending into the first quadrant, is constructed next. This intersects the vertically upward v-axis at P_6 and determines $v = v_0$, again of no special interest to us. The intersection of the lines $P_5 P_6$ and $P_3 P_4$ occurs at P_7 in the first quadrant and this point determines the value $R = R_0$ which we are seeking. If the point P_7 does not lie on one of the contours shown on the nomogram then R_0 can be interpolated from its proximity to neighbouring R-contours. Alternatively, the user may determine the ratio of $\frac{OP_4}{OP_6}$, or $\frac{u_0S_0}{v_0}$, that is, the "slope" of the line OP_7, which corresponds to R_0.

In the nomogram of Figure 1, the various lines have been drawn for $i^*_0 = 6$, $m_0 = 2$, $N_0 = 24$, and $S_0 = 10\,000$. The value of R_0 is found from the nomogram to be approximately \$195; the exact value obtained from (1.6) is, correct to the nearest cent, R_0 = \$193,95. Naturally, the accuracy of the nomogram is restricted by its size, the accuracy in drawing the various lines and the thickness of one's pencil. Reasonable care on a moderately sized nomogram readily gives accuracy to within one or two per cent.

4. Sensitivity of R with respect to m

We now return to discuss the effect of m, the number of repayments per adjustment period, on the amount of the repayment instalment. It follows immediately from equations (1.6) and (1.7) that

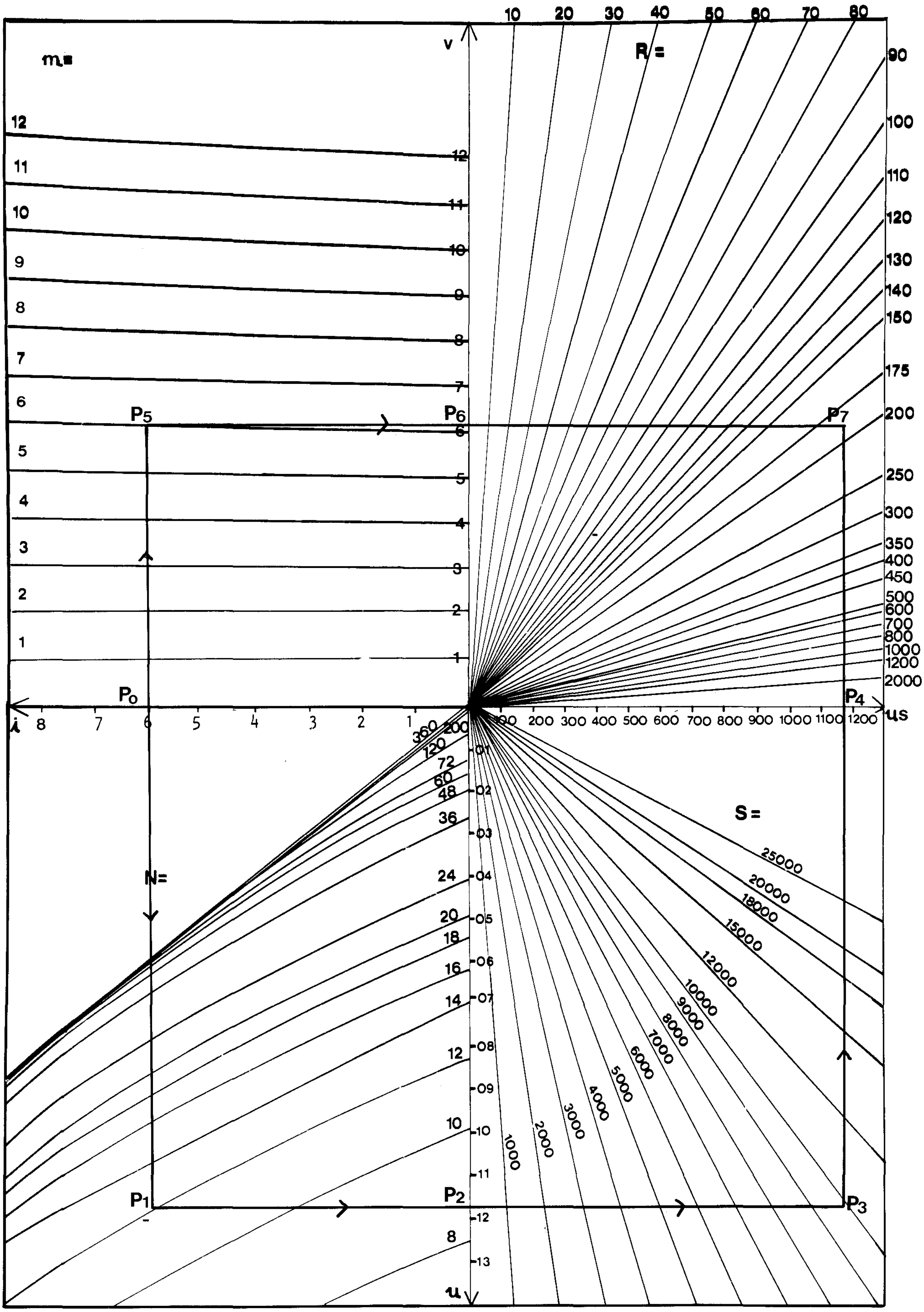
$$R = \frac{R^*}{m + \frac{1}{2}(m - 1)i^*} = \frac{R^*}{v} \qquad (4.1)$$

and hence that $mR \leq R^*$ since $i^* \geq 0$ and $m \geq 1$. Further, it is apparent that $\frac{mR}{R^*}$ decreases as m increases and, of course, increases as m decreases. In fact, it is clear from (4.1) that $\frac{R^*}{mR}$ is an increasing function of m; this could have been inferred directly from the form of the v-contours in the second quadrant of the nomogram in Figure 1. Of course, the above comments are also intuitively clear; we would expect, ceteris paribus, the repayment instalments to be smaller the more frequently they are required! It is of interest to examine the ratio $\frac{R^*}{mR}$ in relation to unity. Clearly, from (4.1) $\frac{R^*}{mR} = 1 + \frac{1}{2}(1 - \frac{1}{m})\ i^*$ and hence $\frac{R^*}{mR}$ increases as m increases and decreases as m decreases. It, therefore, follows that a borrower using a rested interest loan will pay less in total repayments the more repayments are required per adjustment period. However, whether or not this is to the borrower's advantage will depend on market rates of interest in relation to the loan and, in particular, at what rate the borrower can invest repayment monies not yet due in the marketplace. For analysis of these considerations, the reader is referred to Rickard and Russell.[8] We now turn our attention to the question of what, exactly, is the true cost of borrowing for a rested interest loan with $m > 1$.

5. The true cost of a "rested interest" loan

In order to determine the real or true cost associated with a "rested interest" loan we will determine the *effective rate of interest,* 100e% per annum, applicable to the loan and, in addition, the *equivalent nominal rate of interest per repayment period,* $100i^*_E$%. The latter we define to be the nominal rate corresponding to an identical loan (the same loan duration, repayment frequency, loan amount and repayment instalment) with *simultaneous repayment and adjustment of principal.* Clearly, the nominal interest rate per annum corresponding to i^*_E per repayment period is given by $i_E = mni^*_E$, since there are mn repayments per annum. On putting $m = 1$ in (1.6) and replacing n by mn it readily follows that

Figure 1: Nomogram for determining the repayments of rested loans



$$\frac{i_E^*(1+i_E^*)^M}{(1+i_E^*)^M-1} = \frac{i^*(1+i^*)^N}{[m+\frac{1}{2}(m-1)i^*][(1+i^*)^N-1]}, \quad (5.1)$$

where, for convenience, we have written $M = mny = mN$. We wish to determine i_E^* (and hence $i_E = mni_E^*$) given m, N (and hence M) and i^*. We also wish to determine the effective rate of interest, 100e% per annum, applicable to the loan. This is given by

$$1 + e = (1 + i_E^*)^{mn}$$

or

$$e = (1 + i_E^*)^{mn} - 1. \quad (5.2)$$

It is not possible to express i_E^* or e explicitly in terms of i^*. It is possible, however, to obtain i_E^*, and hence e, iteratively from (5.1).

Finally, it is of interest to observe how i_E and i_E^* behave with increasing m. In Figures 2, 3 the graphical behaviour is depicted, and it is immediately clear that both i_E and i_E^* decrease steadily with increasing m. Note that for the convenience of presentation the graphs in Figures 2, 3 are continuous, rather than being a discrete set of points corresponding to the integral values of m.

Figure 2

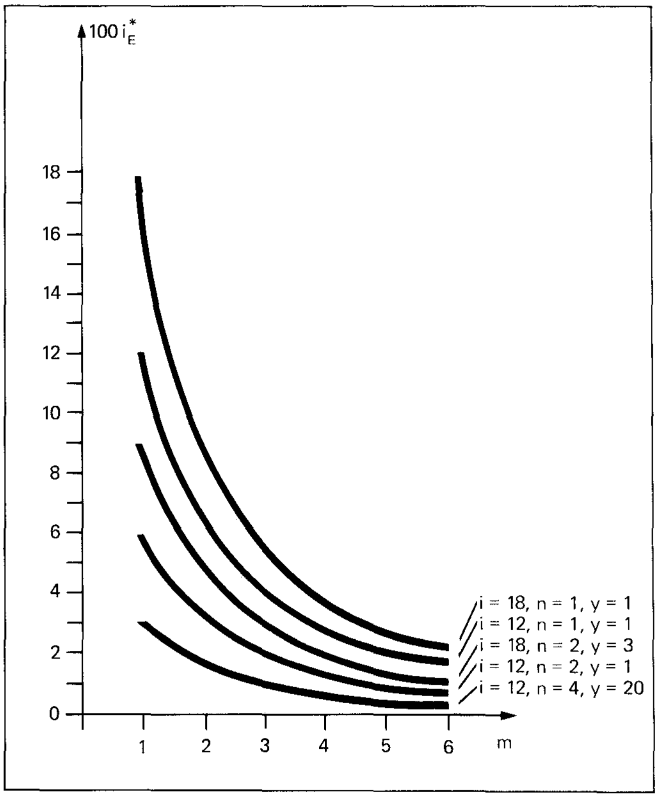
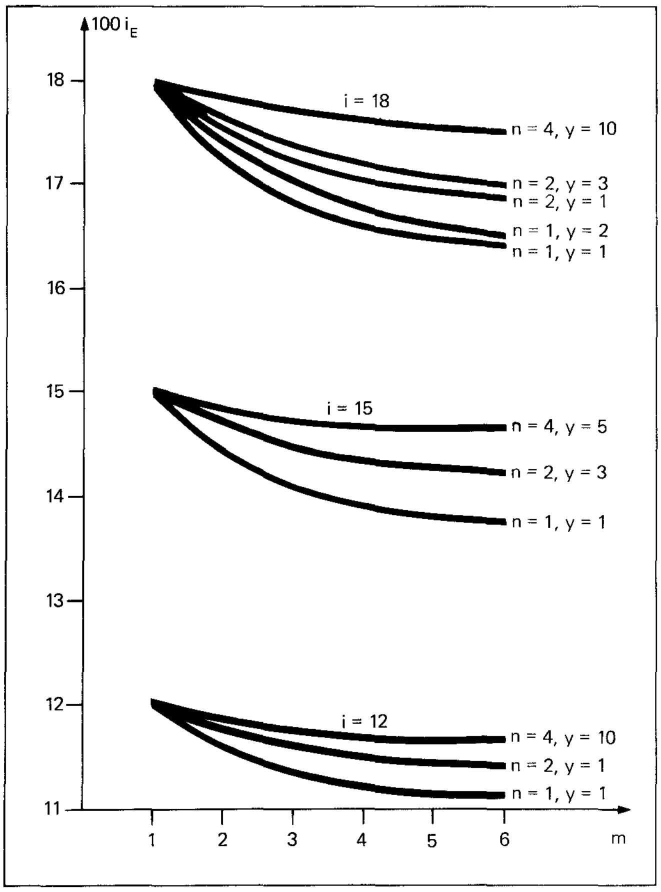


Figure 3



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