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Constrained leverage in a benchmark-relative framework

1. INTRODUCTION

The Mean-Variance (MV) approach to portfolio selection has provided rational investors with a framework for under uncertainty. The work of Markowitz, Tobin and Sharpe led to a better appreciation of how investments combine in a risk and reward sense. Tobin's contribution in particular simplified the portfolio selection decision. His Separation Theorem reduced the portfolio selection problem to an allocation between the market portfolio and the risk-free rate according to the investor's individual risk aversion.

Many modern portfolio managers however are concerned with the restrictions placed on them by their investment mandates and have adopted a relative framework for portfolio selection to replace the traditional mean-variance approach. The use of tracking error (TE)[1] or return relative to a pre-selected or mandated benchmark (see for example Franks (1992), Chow (1995), Roll (1992) and Clarke, Krase and Statman (1994)) has led to the replacement of variance with tracking error variance (TEV) as a measure of uncertainty.

In this relative framework, Tobin's Separation Theorem and the status of the risk-free asset have to be reconsidered. In the MV framework, the risk-free asset (i.e. cash or equivalent fixed interest instrument) has no risk (variance) and zero correlation with any risky asset and thus combines linearly with any of these risky assets in MV space. However, in a relative framework, the risk-free asset is very risky (high TEV) since its returns are divergent from a benchmark of risky-assets. Thus the only truly "risk-free" asset in this benchmark-relative framework of portfolio selection is the benchmark itself. Consequently, the risk-free rate is no longer risk-free in a benchmark-relative sense and Tobin's Separation Theorem no longer applies in the same way.

Two major consequences of the high-"risk" status of the risk-free rate in a relative portfolio selection framework are discussed in this article. Firstly, in a relative framework, the risk-free asset is no longer a certainty equivalent but a substantially sub-optimal investment[2]. In fact, when active managers are constrained to invest a proportion of their portfolio in the risk-free asset, their efficient set of choices becomes increasingly sub-optimal. This concept is demonstrated empirically in the first section of this article.

Secondly, managers cannot gear their fund linearly along the same relative risk - relative return trade-off by investing in or borrowing at the risk-free rate, as described by the Separation Theorem in the traditional risk-return framework. Instead, fund managers who seek to gear their fund in TE/TEV space but maintain the same relative return (TE) – relative risk (TEV) trade-off must borrow/lend the benchmark. The second section of this article shows that the TEV or "active" analogy of constrained leverage in a MV framework is constrained passive management. This section contains analysis and proofs of the intuitive result that active managers who are subject to greater constrained passive management will choose to incur more active risk in the remaining fund than they would if they were unconstrained.

2. CONSTRAINED LENDING AND BORROWING IN A BENCHMARK-RELATIVE FRAMEWORK

2.1 Data and analysis

By way of demonstration and to represent the investment environment available to South African investors, the universe of assets considered in this article consists of 53 Johannesburg Stock Exchange (JSE) Actuaries Indices, one of which is the All Share Index (JOHMKT) which is used as a benchmark. A summary of the data is recorded in the appendix. The overnight interest rate at the time of this study was 10.10% NACQ[3] (about 0.84% per month) and this rate is used as a proxy for the risk-free rate in this study.

For the purpose of the discussion in this section, let us assume that the hypothetical investors discussed here can be represented by an exponential utility function on wealth or, in this case, relative wealth[4]. The TE-TEV optimal unconstrained choices for active

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[1]Tracking error in this article refers to the return relative to a benchmark portfolio, whereas tracking error variance refers to the variance of these relative returns.

[2]It can happen that the risk-free rate outperforms risky assets, such as in a market crash or correction. However, over the longer term, investors expect to be rewarded for bearing risk and consequently the risk-free asset is expected to underperform risky assets. In either case the risk-free asset remains a high TEV investment.

[3]Nominal Annual Compounded Quarterly (October 2000).

[4]See Kallberg and Ziemba for alternative forms of utility functions in the finance context.

managers with different aversions to TEV are plotted along the efficient frontier in Figure 1[5] (the coordinates of these choices in TE/TEV space are recorded in Table 1).

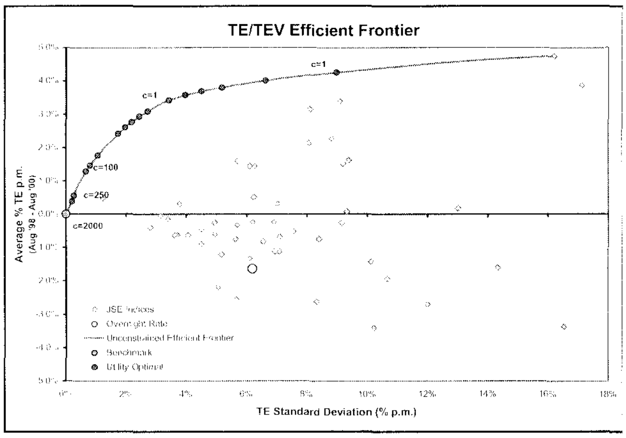

Figure 1: Utility-maximising investments for unconstrained active managers of different TEV-aversion

In the case of this particular investment environment with these given parameters, only an active manager with an extremely high aversion to TEV (c = 2000) would elect to hold the benchmark itself thereby adopting a passive strategy. The smaller the active manager's TEV aversion (smaller c), the more "active" their optimal strategy and the greater their investment's distance from the benchmark in TE/TE standard deviation space. Also notice in Figure 1, the unfavourable position of the risk-free asset. No rational, unconstrained, TEV-averse manager would select the risk-free rate as part of their portfolio because of its high TEV and negative TE.

2.2 Constrained investment in the risk-free rate

The Separation Theorem's assumption of limitless borrowing or lending seldom applies to institutional investors. Many publicly listed funds are not permitted to gear up their returns and risks (i.e. to borrow) and have constraints placed on their investment in "risk-free" instruments. For example, the Unit Trusts Control Act No 54 of 1981 constrains all unit trust funds to hold at least 5% of the value of their portfolio in liquid assets[6]. The spirit of this restriction is to have at least 5% of the fund in assets which can be liquidated quickly and easily without risk to their price or return in order to provide for fund redemptions.

This restriction is at odds with the active manager who typically (albeit unfairly) has a fully invested benchmark and would ideally invest the full value of the fund in assets of the same class as that benchmark.

Figure 2 demonstrates the disadvantages associated with a constrained investment in the risk-free rate in an active management framework.

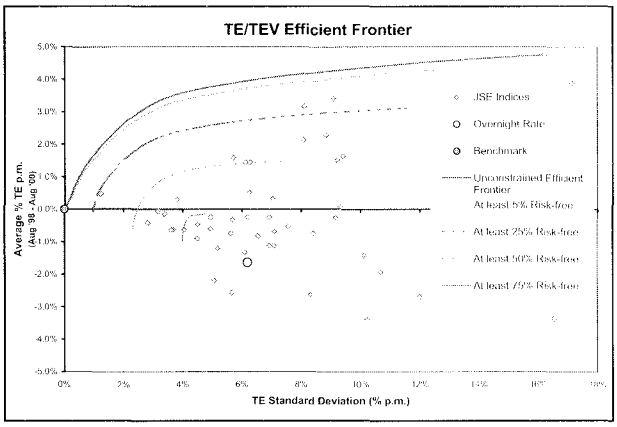

Figure 2: TE/TEV efficient frontier with constrained investment in the risk-free asset

The efficient set of investments move further southeast on the TE/TEV framework with increasing constrained investment in the risk-free asset as a consequence of the high TEV and the negative TE of the risk-free asset. Thus the segment of the TE/TEV efficient frontier that represents low TEV-aversion will become less risky with increasing investment in the risk-free rate but the segment representing high TEV aversion will become more risky. Notice that, under constrained investment in the risk-free rate, there is no passive alternative and the benchmark itself cannot be held or mimicked with 100% of the funds. Therefore, the constrained efficient frontier stops short of the "riskless" position in this framework, namely the benchmark. The more risky (in the MV sense) the benchmark and the greater its excess returns (over-and-above the risk-free rate), the more dramatic the effect of a constrained investment in the risk-free rate will be in a TE/TEV paradigm.

Table 1 comprises the co-ordinates of the utility optimal portfolios with and without a constrained investment in the risk-free rate. In contrast to the effects of such a constraint in a MV framework, the efficient frontiers under a constrained investment in cash are not linear projections of each other and the risks of the selected portfolios are not uniformly affected in one direction (refer to Bradfield and Raubenheimer (2001)). That is, a constrained investment in cash does not necessarily mean that an active manager should take on greater TEV.

[5] Refer to the Appendix for details on the optimisation procedure.

[6] Section 6 number 2 of the Unit Trusts Control Act No. 54 of 1981 states, "Every management company, other than a management company in property shares, shall include in every unit portfolio liquid assets with an aggregate market value of not less than five per cent of the aggregate market value of all the assets comprised in the unit portfolio..."

Table 1: Utility-Maximising Portfolios for Unconstrained and Constrained Investment in the Risk-Free Asset

Risk Aversion (c)		1	2	3	4	5	10	20	25	30	40	50	100	150	250	1000	2000
No	**TE Std Dev**	9.00	6.64	5.19	4.52	3.97	3.43	2.72	2.45	2.19	1.97	1.74	1.08	0.82	0.68	0.26	0.00
constraints	**Std Dev**	12.66	10.04	8.66	8.02	7.45	7.07	6.78	6.66	6.55	6.46	6.36	6.23	6.19	6.18	6.16	6.20
	TE	4.25	4.01	3.80	3.69	3.58	3.42	3.09	2.93	2.77	2.62	2.42	1.77	1.47	1.29	0.55	0.00
5% in the	**TE Std Dev**	8.83	6.42	5.14	4.46	3.90	3.35	2.69	2.41	2.16	1.93	1.72	1.05	0.79	0.67	0.27	0.24
Risk Free	**Std Dev**	12.45	9.77	8.50	7.84	7.25	6.84	6.58	6.46	6.36	6.27	6.23	6.16	6.14	6.13	6.12	6.12
Asset	**TE**	3.99	3.74	3.56	3.45	3.34	3.18	2.88	2.73	2.57	2.41	2.23	1.59	1.31	1.16	0.40	0.30

In this particular example, active managers with high aversion to TEV (c = 2000), would have chosen a fund with more TEV under a constrained investment in the risk-free rate since the TEV-free option of investing only in the benchmark is not available under these constraints. By contrast, less TEV-averse managers would have selected a fund with less TEV under a constrained investment in the risk-free rate simply because of the shape and the shrinkage of the efficient frontier towards the risk-free rate. Ultimately the risk-free rate in a TE/TEV framework behaves much like a badly performing risky asset in a traditional framework and contributes negatively to the TE of a fund.

3. "LEVERAGE" IN A BENCHMARK-RELATIVE FRAMEWORK

3.1 Leverage using the benchmark

The true "risk-free" asset in an active management sense is the benchmark itself. Thus if a proportion, Y, is invested in the benchmark and the remainder is invested in any risky asset in the TEV framework, the combination is linear in active reward (TE) – active risk (TE standard deviation) space.

$$
\begin{aligned}
TE_p &= YR_{benchmark} + (1-Y)R_{risky-asset} - R_{benchmark} \\
&= (1-Y)\left(R_{risky\ asset} - R_{benchmark}\right) \\
&= (1-Y)TE_{risky-asset}
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{TEV_p} &= \sqrt{Var\left[(1-Y)TE_{risky-asset}\right]} \\
&= |1-Y|\sqrt{TEV_{risky\ asset}}
\end{aligned}
$$

Thus, in a relative or active management portfolio selection framework, a fund/asset can be levered along a line connecting the benchmark (the origin) with the non-levered fund/asset by borrowing (Y<0) or lending (Y>0) the benchmark to any extent. That is:

$$
\left(\sqrt{TEV_{risky\ asset}}, TE_{risky\ asset}\right) \xrightarrow{\text{levered}} \left(|1-Y|\sqrt{TEV_{risky-asset}}, (1-Y)TE_{risky\ asset}\right)
$$

Of course the risk-free asset also combines in this way with the benchmark:

$$
\left(\sqrt{TEV_{risk-free}}, TE_{risk-free}\right) = \left(\sigma_{benchmark}, -\left(R_{benchmark} - R_f\right)\right) \xrightarrow{\text{levered}} \left(|1-Y|\sigma_{benchmark}, (Y-1)\left(R_{benchmark} - R_f\right)\right)
$$

Thus if, for a particular portfolio, less than 100% of funds are invested in the benchmark (i.e. $0 < Y \le 1$) and the remainder is invested in the risk-free rate, the portfolio will have a negative TE and increasing TEV with increasing investment in the risk-free rate. However, if some funds are borrowed at the risk-free rate and all funds are invested in the benchmark (i.e. Y > 1), the fund will be geared up in TE-TE standard deviation space in the same linear way as in MV space.

Investors are seldom able to hold assets short and borrowing is usually only available at the risk-free rate. However, one can still gear up in TE-TE standard deviation space without holding any assets short by splitting the fund between the benchmark and any other asset. Figure 3 depicts the gearing possibilities in TE-TE standard deviation space.

A, B, C and D are hypothetical assets in the investment universe. Figure 3 shows that these assets combine linearly with the benchmark. If the benchmark is the tangent portfolio, the two straight lines between the origin (benchmark) and the risk-free rate and the origin and the "Long Benchmark & Short Risk-Free Rate" represent the CML and will offer the highest TE/TEV ratio and consequently will have the steepest slopes in this framework. However, even if the benchmark is not the tangent portfolio, it still combines linearly with the risk-free rate and all other risky assets although its combination with the risk-free rate may not necessarily provide the greatest TE/TEV ratio in the investment universe. Figure 3 shows that the combination of the benchmark with a poorly performing hypothetical asset D can provide greater gearing than the risk-free rate. If any asset which underperforms the benchmark may be held short (i.e. borrowed), the resultant set of combinations of the benchmark and this asset will form a positively sloping component as shown in Figure 3. Conversely, holding this same asset long (lending) will incur negative TE and large TEV. The downward sloping component in Figure 3 makes this clear. Lastly, the curved lines between assets A, B and C demonstrate the non-linear relationship between risky assets and the risk-free asset in an active management framework whereas, in the traditional MV framework, these curved lines would be straight and positively sloped.

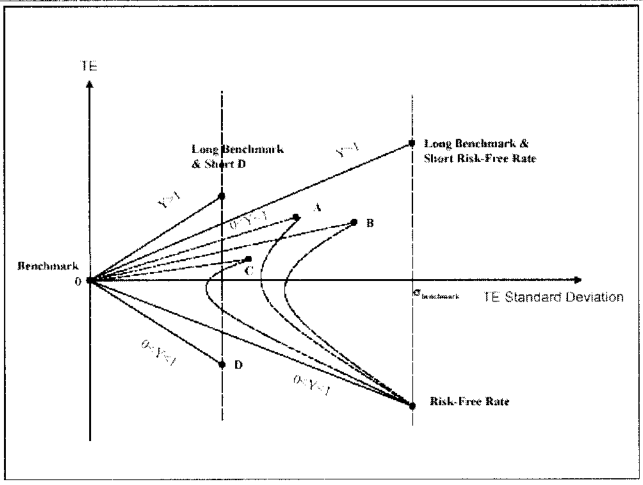

Figure 3: Gearing in TE-TE standard deviation space

3.2 TE-TEV utility optimisation under different passive management constraints

Tobin's Separation Theorem under the assumption of limitless borrowing and lending at the risk-free rate and the developments of Bradfield and Raubenheimer (2001) regarding constrained investment in the risk-free rate have an analogy when it comes to constrained leverage in an active management framework. That is, the greater the extent of constrained lending (borrowing), the greater (lower) the risk of the remaining unconstrained utility-optimal portfolio. However, in this active framework, leverage is attained via the benchmark rather than the risk-free rate. It would thus be more accurate to refer to leverage in a TE/TEV framework as constrained passive management. To constrain fund managers to invest in the benchmark itself is to constrain a proportion of funds under their management to be managed passively.

Therefore, just as an investor should select a higher risk portfolio when bound by a constraint to invest a greater proportion in the risk-free rate, an active manager who is bound by a constraint to manage a greater proportion of their fund passively should take more active risk on the remainder of the fund. This presumes that the active manager's utility is a function of TE and TEV rather than the utility function of return and variance.

The expected utility of a risk-averse benchmark-sensitive manager can be defined as follows:

$$U\left(\sqrt{TEV}, TE\right) = v(TE) - w\left(\sqrt{TEV}\right)$$

where v(TE) is concave and $w\left(\sqrt{TEV}\right)$ is convex reflecting a rational active manager's attitude to relative return and relative risk respectively[7].

The proof that the greater the portion of the fund which is managed passively, the greater the active risk of the active portion of the fund should be, follows.

Since $w\left(\sqrt{TEV}\right)$ is convex, $w''\left(\sqrt{TEV}\right) > 0$ by definition and so $w'\left(\sqrt{TEV}\right)$ is increasing. Similarly, since $v(TE)$ is concave, $v''(TE) < 0$ and $v'(TE)$ is decreasing.

Since *U* is an implicit function of $\sqrt{TEV}$ and TE, the gradient of the relationship between TE and $\sqrt{TEV}$ along the iso-utility line is as follows:

$$\frac{dTE_{Utility}}{d\sqrt{TEV}_{Utility}} = \frac{w'\left(\sqrt{TEV}\right)}{v'(TE)}$$

Thus the slopes of the iso-utility lines are steeper at ($\sqrt{TEV_2}$; TE_2) than ($\sqrt{TEV_1}$; TE_1) if $\sqrt{TEV_1} < \sqrt{TEV_2}$ and $TE_1 < TE_2$ since:

$$\frac{w'\left(\sqrt{TEV_1}\right)}{v'(TE_1)} < \frac{w'\left(\sqrt{TEV_2}\right)}{v'(TE_2)}$$

Therefore:

$$\left.\frac{dTE_{Utility}}{d\sqrt{TEV}_{Utility}}\right|_{\sqrt{TEV_1}:TE_1} < \left.\frac{dTE_{Utility}}{d\sqrt{TEV}_{Utility}}\right|_{\sqrt{TEV_2}:TE_2} \qquad \ldots (4)$$

[7]The fact that this expected utility function is additive relies on the assumption that the two attributes with which we are concerned satisfy the corresponding trade-offs condition. See Keeney and Raiffa (1976 (page 91)). See also Clarke, Krase and Statman (1994) for an example of the use of such a utility function.

The expected utility of a risk-averse investor is represented by indifference curves U_1, U_2 and U_3 in Figure 4.

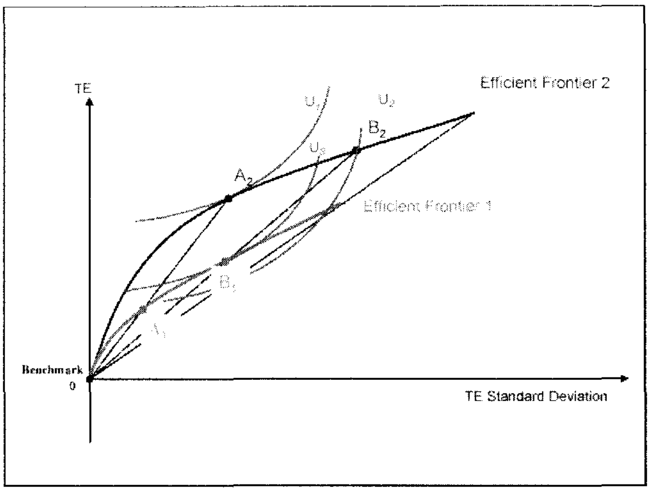

Figure 4: Constrained efficient frontiers and utility functions

Let efficient frontiers 1 and 2 be two passive management-constrained efficient frontiers (as shown in Figure 4) with frontier 2 constrained to hold a smaller proportion, Y_2, in the benchmark than the proportion, Y_1, held by frontier 1 i.e. frontier 2 is less constrained than frontier 1. In particular, for any point ($\sqrt{TEV_1}$;TE_1) on efficient frontier 1, there is a corresponding point ($\sqrt{TEV_2}$;TE_2) on efficient frontier 2 with co-ordinates ($K\sqrt{TEV_1}$, KTE_1) where $K>1$ is the factor[8] by which frontier 1 has been constrained relative to frontier 2. Consequently for any two corresponding points on frontier 1 and 2, $\sqrt{TEV_1} < \sqrt{TEV_2}$ and $TE_1<TE_2$. It follows (from the result in (4)) that the slope of the iso-utility line at any point on frontier 2 is steeper than the slope of the iso-utility line at a corresponding point on frontier 1.

Let us assume that A_2 is the utility-maximizing portfolio along frontier 2. This means that the iso-utility curve is tangent to the efficient frontier at A_2 and hence the slope of the iso-utility function is the same as the slope of the efficient frontier at A_2. But the slope of the efficient frontier at A_2 is also the same as the slope of efficient frontier 1 at A_1 (refer to the Appendix).

Therefore:

$$\left.\frac{d\mu_{\text{Utility}}}{d\sigma_{\text{Utility}}}\right|_{\sigma_{A2};\mu_{A2}} = \left.\frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}}\right|_{\sigma_{A2};\mu_{A2}} = \left.\frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}}\right|_{\sigma_{A1};\mu_{A1}}$$

[8] $K=(1-Y_2)/(1-Y_1)$ where $0< Y_2< Y_1<1$ and thus $K>1$.

We also know that the slope of the iso-utility line at A_2 is steeper than the slope of the iso-utility line at A_1. Thus the slope of the efficient frontier at A_1 is also steeper than the slope of the iso-utility line at A_1:

$$\left.\frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}}\right|_{\sigma_{A1};\mu_{A1}} > \left.\frac{d\mu_{\text{Utility}}}{d\sigma_{\text{Utility}}}\right|_{\sigma_{A1};\mu_{A1}} = -\left.\frac{\partial U}{\partial \sigma}\frac{\partial \mu}{\partial U}\right|_{\sigma_{A1};\mu_{A1}}$$

Rearranging the terms:

$$\left.\frac{d\mu_{\text{Eff frontier}}}{d\sigma_{\text{Eff frontier}}}\right|_{\sigma_{A1};\mu_{Aq}} + \left.\frac{\partial U}{\partial \sigma}\frac{\partial \mu}{\partial U}\right|_{\sigma_{A1};\mu_{Aq}} > 0$$

Multiplying through by $\left.\frac{\partial U}{\partial \mu}\left(d\sigma_{\text{Eff frontier}}\right)\right|_{\sigma_{Aq};\mu_{A1}}$ we obtain the following result:

$$\left.\frac{\partial U}{\partial \mu}\left(d\mu_{\text{Eff frontier}}\right)\right|_{\sigma_{A1};\mu_{A1}} + \left.\frac{\partial U}{\partial \sigma}\left(d\sigma_{\text{Eff frontier}}\right)\right|_{\sigma_{A1};\mu_{A1}} > 0$$

The left-hand side of this inequality is the inner product of two paths: a) the normal to the expected utility function and b) a path along the efficient frontier in the direction of positive return (μ) and positive risk (σ). The fact that this product is positive means that the expected utility increases as one moves up efficient frontier 1 from A_1 in the direction of increasing return and increasing risk (north-east in TE/TE standard deviation space). This increase implies that, although portfolio A is the utility optimal portfolio of actively managed assets along frontier 2 for our hypothetical manager, when the same manager is confined to invest along frontier 1, there exist portfolios of higher utility than A in a higher active risk (TEV) position, such as portfolio B_1.

To conclude, the higher an active manager's constrained investment in the benchmark, the higher the risk ($\sqrt{TEV}$) of this manager's optimal remaining portfolio is and conversely, the lower an active manager's constrained investment in the benchmark, the lower the risk ($\sqrt{TEV}$) of this manager's optimal remaining portfolio is. Thus, intuitively, the utility-optimal remedy to increased constrained passive management of a fund is to increase the risk of the active component of the same fund.

3.3 Empirical example of constrained investment in the Benchmark

This section illustrates the effect of constrained passive management on the active manager. It is not uncommon for fund mangers to be instructed to hold a certain proportion of assets under management in passive portfolios which emulate the benchmark and to focus their active management skills on the remainder of the fund. Managers who have as a benchmark a published market index will usually be able to invest a proportion of their fund in securities which mimic the benchmark exactly such as exchange traded funds (ETF's) or futures and options. For more complex benchmarks, an index tracker has to be designed.

The previous section concluded that the greater the proportion of a fund which is constrained to be invested in a passive style, the more active risk the remainder of the fund should take. Figure 5 shows an example of these constraints to manage 80%, 60%, 40% and 20% of a fund passively by constraining the investment in the benchmark to be at least 20%, 40%, 60% and 80% respectively. Table 2 lists the co-ordinates of the optimal unconstrained portfolios, the optimal constrained portfolios and the active segment of the constrained portfolios for various levels of TEV aversion and a 20% minimum constrained investment in the benchmark.

Table 2: Utility-Maximising Portfolios for Unconstrained and Constrained Investment in the Benchmark

Risk Aversion (d)		1	2	3	4	5	10	20	30	40	50	100	150	250	1000	2000
No	**TE Std Dev**	9,00	6,64	5,19	4,52	3,97	3,43	2,72	2,19	1,97	1,74	1,08	0,82	0,68	0,26	0,00
constraints	**TE**	4,25	4,01	3,80	3,69	3,58	3,42	3,09	2,77	2,62	2,42	1,77	1,47	1,29	0,55	0,00
> 20% in	**TE Std Dev**	8,34	5,71	4,76	4,01	3,38	2,99	2,45	1,96	1,71	1,58	1,02	0,76	0,63	0,25	0,00
Benchmark	**TE**	3,50	3,26	3,13	3,02	2,91	2,81	2,62	2,35	2,19	2,09	1,58	1,29	1,14	0,53	0,00
Active	**TE Std Dev**	10,42	7,14	5,94	5,01	4,22	3,73	3,07	2,45	2,13	1,97	1,28	0,95	0,79	0,31	0,00
Segment	**TE**	4,38	4,07	3,91	3,77	3,63	3,51	3,27	2,93	2,73	2,62	1,98	1,62	1,43	0,66	0,00

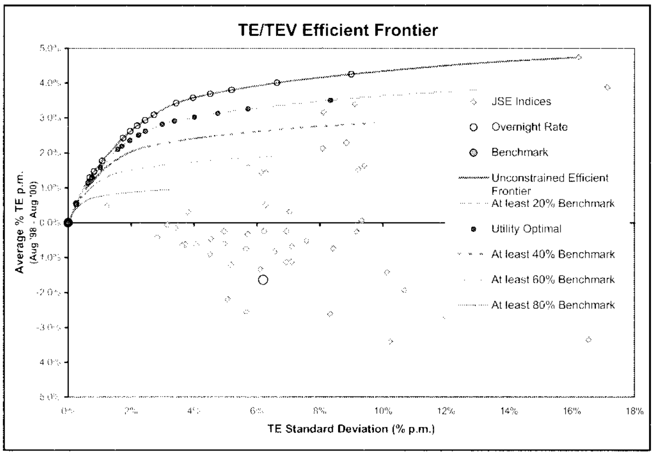

Figure 5: Efficient frontier with constrained investment in the benchmark

It is clear from the chart that greater constrained investment in the benchmark, and thus greater investment in an asset without risk, will decrease the overall risk of the portfolios chosen by active managers. However, the active component of these portfolios (that part of the portfolio which is not invested in the benchmark) will have greater TEV than the same manager's unconstrained portfolio. The only exception to this pattern is an active manager who would under unconstrained conditions invest the same constrained amount passively and who is thus effectively unbound by the constraint, such as, in this example, the manager with risk aversion c=2000.

4. CONCLUSIONS

This article extends the Separation Theorem and the findings of Bradfield and Raubenheimer (2001) into the TE/TEV paradigm. The analysis here shows that active managers cannot gear their fund linearly along the same active risk – active return trade-off by investing in or borrowing at the risk-free rate. To the contrary, constrained investment in the risk-free rate will force the portfolio into less efficient TE/TEV space. Instead, portfolio managers who seek to gear their fund in TE/TEV space but maintain the same relative return (TE) – relative risk (TEV) trade-off must borrow/lend the benchmark.

While borrowing the benchmark (i.e. selling the benchmark short) is a practice which is usually beyond regulatory and legislative restrictions, it is possible and common to constrain a long investment in the benchmark. This strategy translates into a constraint on active management by constraining the manager to invest a portion of the fund passively. This analysis shows that the greater the constrained investment in the benchmark, when this constraint is binding, results in optimising the active manager's utility for the active segment of the fund at a higher active risk level.

APPENDICES

Data Summary

The historical month-end, last-traded prices of various indices on the Johannesburg Stock Exchange (JSE) were downloaded from Bloomberg for a two year period (August 1998 to August 2000). Table 3 below summarises the active and passive returns and risks of these indices over the period which were used to generate the efficient frontiers in this article.

The first three columns comprise each index's Bloomberg "Ticker" code, the arithmetic average of the monthly returns, and the standard deviation of these monthly returns. The risk-reward ratio is in the fourth column. The following three columns comprise the average monthly return less that of the benchmark (JOHMKT) i.e. the TE of each index, the standard deviation of these TE's and the ratio of TE to TE standard deviation from largest (rank of one) to smallest (rank of 54).

Table 3: Summary statistics of the empirical investment universe

Index	Average Return (% p.m.)	Std Dev (% p.m.)	*Ranked Ret/ Std Dev Ratio*	Average TE (% p.m.)	TE Std Dev (% p.m.)	*Ranked TE/ TE Std Dev Ratio*
JOHPPER Index	6,3%	19,7%	***17***	4,7%	16,2%	***4***
JOHDEVEL Index	6,0%	18,7%	***16***	3,9%	17,1%	***10***
JOHPLAT Index	6,0%	9,0%	***1***	3,4%	9,1%	***3***
JOHMHOUS Index	4,8%	11,7%	***9***	2,1%	8,1%	***6***
JOHINVTR Index	4,8%	10,6%	***4***	3,2%	8,1%	***2***
JOHDIAM Index	4,4%	11,0%	***10***	2,3%	8,8%	***7***
JOHDIND Index	4,3%	6,9%	***2***	1,6%	5,7%	***5***
JOHMIN Index	4,3%	9,3%	***3***	1,4%	6,1%	***8***
JOHRES Index	4,2%	9,5%	***8***	1,4%	6,3%	***9***
JOHNMIN Index	3,6%	12,6%	***20***	1,6%	9,4%	***11***
JOHCHEM Index	3,5%	12,1%	***19***	1,5%	9,2%	***12***
JOHALSI Index	3,0%	6,7%	***5***	0,5%	1,2%	***1***
JOHOMETL Index	2,7%	9,3%	***18***	0,3%	7,0%	***15***
JOHINDI Index	2,7%	6,7%	***12***	0,3%	3,8%	***14***
JOHMKT Index	***2,5%***	***6,2%***	***11***	***0,0%***	***0,0%***	
JOHRET Index	2,5%	9,7%	***22***	0,5%	6,3%	***13***
JOHPROPT Index	2,4%	5,3%	***6***	-0,2%	6,2%	***21***
JOHSTEEL Index	2,4%	15,4%	***36***	0,2%	13,0%	***16***
JOHFIND Index	2,3%	6,9%	***14***	-0,1%	3,2%	***18***
JOHLOAN Index	2,2%	4,9%	***7***	-0,2%	6,9%	***20***
JOHBANK Index	2,2%	8,7%	***25***	-0,2%	4,9%	***23***
JOHINDST Index	2,1%	6,3%	***15***	-0,1%	3,4%	***22***
JOHINFO Index	2,1%	11,4%	***32***	0,1%	9,3%	***17***
JOHHEA Index	2,0%	8,2%	***26***	-0,5%	7,6%	***25***
JOHLINS Index	1,9%	7,9%	***24***	-0,6%	4,1%	***35***
JOHREAL Index	1,9%	5,0%	***13***	-0,7%	7,1%	***27***
JOHFINID Index	1,9%	6,7%	***21***	-0,4%	2,8%	***34***
JOHINS Index	1,9%	7,4%	***23***	-0,6%	3,6%	***40***
JOHPKPR Index	1,8%	10,6%	***33***	-0,3%	9,2%	***19***
JOHGOLD Index	1,8%	13,1%	***39***	-1,9%	10,7%	***41***
JOHSINS Index	1,7%	7,4%	***27***	-0,3%	5,7%	***24***
JOHFINC Index	1,7%	7,9%	***30***	-0,6%	3,7%	***38***
JOHFINCL Index	1,7%	7,9%	***30***	-0,6%	3,7%	***38***
JOHINDC Index	1,6%	7,6%	***29***	-0,5%	4,5%	***28***
JOHFOOD Index	1,5%	6,4%	***28***	-0,6%	5,0%	***30***
JOHBFL Index	1,4%	8,5%	***34***	-0,9%	4,5%	***42***
JOHBUILD Index	1,3%	8,1%	***35***	-1,1%	7,1%	***36***
JOHFH Index	1,2%	10,7%	***41***	-0,7%	8,4%	***26***
JOHELEC Index	1,2%	7,8%	***37***	-0,7%	5,6%	***32***

JOHSERV Index	1,1%	7,4%	***38***	-1,2%	5,2%	***46***
JOHBEV Index	1,0%	9,1%	***42***	-1,1%	6,9%	***37***
Overnight Rate	***0,8%***	***0,0%***		***-1,8%***	***6,3%***	***47***
JOHBHL Index	0,6%	4,9%	***40***	-1,3%	6,1%	***44***
JOHCOAL Index	0,6%	13,5%	***43***	-1,6%	14,3%	***29***
JOHDCM Index	0,2%	11,2%	***44***	-1,4%	10,1%	***33***
JOHPRIV Index	0,0%	9,9%	***45***	-0,8%	6,6%	***31***
JOHPROP Index	0,0%	7,0%	***46***	-2,6%	8,3%	***48***
JOHLEIS Index	-0,3%	7,1%	***47***	-2,2%	5,1%	***51***
JOHFINS Index	-0,7%	9,4%	***48***	-2,6%	5,7%	***52***
JOHCO Index	-1,2%	8,4%	***51***	-3,4%	10,2%	***49***
JOHTELE Index	-1,4%	12,5%	***50***	-2,7%	12,0%	***45***
JOHEXP Index	-1,6%	15,6%	***49***	-3,4%	16,5%	***43***
JOHREDEV Index	-3,3%	14,2%	***52***	-5,1%	12,5%	***50***
JOHED Index	-4,7%	11,1%	***53***	-6,8%	9,5%	***53***
JOHVC Index	-5,6%	8,3%	***54***	-7,5%	7,5%	***54***

Optimisation procedure

TE-TEV optimisation is used in this analysis. The formula for traditional mean-variance (MV) optimisation is:

Minimise: $-\underline{W}'\underline{R}+\lambda\underline{W}'\Sigma\underline{W}$ for $\lambda=0\ldots\infty$

subject to: (i) $\underline{W'1}=1$

and (ii) $0\le\underline{W}\le1$

where

$\underline{W}$ is the vector of the proportional investment in each asset,

$\underline{R}$ is the vector of returns for each asset,

λ is the risk aversion which ranges from zero (risk-indifferent) to infinity (highly risk-averse)

and

Σ is the variance/covariance matrix of returns.

The two constraints limit (i) the total amount invested to 100% of the portfolio and (ii) short selling and leverage or gearing in any particular asset.

The formula for optimisation of TE and minimising TEV rather than variance is identical in logical structure. In the case of TE-TEV optimisation, a variance/covariance matrix of *relative* returns or TE's $(R_i-R_{benchmark})$ is used instead of a variance/covariance of *absolute* returns (R_i). Thus the formula for TE-TEV optimisation is as follows:

Minimise $-\underline{W}'\underline{TE}+\psi\underline{W}'\Sigma^*\underline{W}$ for $\psi=0\ldots\infty$

subject to $\begin{cases}\underline{W'1}=1\\ 0\le\underline{W}\le1\end{cases}$

Solve for $\underline{W}$

where

$\underline{W}$ is the vector of the proportional investment in each asset,

$\underline{TE}$ is the vector of returns less the benchmark's returns for each asset,

Ψ is the TEV aversion which ranges from zero (TEV-indifferent) to infinity (highly TEV-averse)

and

Σ^* is the variance/covariance matrix of tracking errors. When individual TEV-aversions are discussed (as inTable 1 and Table 2), an exponential utility on (relative) wealth is used. That is, the utility function for individual active managers, *u(w)* can be described as follows:

$$u(w)=1-e^{-cw}\quad c\ge0$$

where

c is the coefficient describing aversion to benchmark-relative risk (TEV). Consequently the indifference or iso-utility curves of this utility function have the following form:

$$U\left(TE,\sqrt{TEV}\right)=cTE-\frac{1}{2}c^2TEV$$

The point on the efficient frontier where this function has its maximum value for any given value of *c* is the optimal investment for the active manager described by this TEV-aversion.

Proof of equal slopes of efficient frontiers at corresponding points

It is show here that the slope of the efficient frontiers 1 and 2 (in Figure 4) at any two corresponding points, ($\sqrt{TEV_1}$; TE_1) and ($\sqrt{TEV_2}$; TE_2), are equal.

Let $E_1(\sqrt{TEV})$ and $E_2(\sqrt{TEV})$ represent frontiers 1 and 2 respectively. Since frontier 2 is levered relative to frontier 1 by a factor K, for two corresponding points ($\sqrt{TEV_1}$; TE_1) and ($\sqrt{TEV_2}$; TE_2) the following relationship holds true:

$$E_2\left(\sqrt{TEV_2}\right) = KE_1\left(\sqrt{TEV_1}\right)$$

Since $\sqrt{TEV_2} = K\sqrt{TEV_1}$:

$$E_2\left(\sqrt{TEV_2}\right) = KE_1\left(\frac{\sqrt{TEV_2}}{K}\right)$$

It follows that

$$\frac{dE_2}{d\sqrt{TEV_2}} = K\frac{dE_1}{d\left(\frac{\sqrt{TEV_2}}{K}\right)}\frac{1}{K} = \frac{dE_1}{d\sqrt{TEV_1}}$$

Thus the slopes of an efficient frontier and its leveraged counterpart at any two corresponding points are the same.

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